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Exact solutions of exactly integrable quantum chains by a matrix product ansatz

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Abstract

Most of the exact solutions of quantum one-dimensional Hamiltonians are obtained thanks to the success of the Bethe ansatz on its several formulations. According to this ansatz, the amplitudes of the eigenfunctions of the Hamiltonian are given by a sum of permutations of appropriate plane waves. In this paper, alternatively, we present a matrix product ansatz that asserts that those amplitudes are given in terms of a matrix product. The eigenvalue equation for the Hamiltonian defines the algebraic properties of the matrices defining the amplitudes. The consistency of the commutativity relations among the elements of the algebra implies the exact integrability of the model. The matrix product ansatz we propose allows an unified and simple formulation of several exact integrable Hamiltonians. In order to introduce and illustrate this ansatz we present the exact solutions of several quantum chains with one and two global conservation laws and periodic boundaries such as the XXZ chain, spin-1 Fateev–Zamolodchikov model, Izergin–Korepin model, Sutherland model, t–J model, Hubbard model, etc. Formulation of the matrix product ansatz for quantum chains with open ends is also possible. As an illustration we present the exact solution of an extended XXZ chain with z-magnetic fields at the surface and arbitrary hard-core exclusion among the spins.

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1. Introduction

The Bethe ansatz [1] and its generalizations emerged over the years as a quite efficient and powerful tool for the exact solution of the eigenspectrum of a great variety of one-dimensional quantum chains and two-dimensional transfer matrices (see, e.g., [2–5] for reviews). According to this ansatz the amplitudes of the wavefunctions are expressed

by a nonlinear combination of properly defined plane waves. Although the question of completeness of the Bethe solutions is in general an open and difficult problem [6, 7], a quantum Hamiltonian is considered exactly integrable if, in the thermodynamic limit, an infinite number of its eigenfunctions are given by the Bethe ansatz.

On the other hand, in the last two decades [8–11], several models were discovered having a ground-state wavefunction exactly known and obtained though an ansatz known as the matrix product ansatz. Differently from the Bethe ansatz, this approach only gives the ground-state eigenfunction, whose amplitudes are expressed in terms of a product of matrices, or more generally in terms of a product of generators of quadratic algebras [12]. In a distinct context, a matrix product ansatz has also been applied quite successfully to the exact solution of the stationary distribution of probabilities of some one-dimensional stochastic models [13]. The similarity between the master equation describing the time fluctuations of these models and the Schrödinger equation in Euclidean times enables us to identify an associated 'quantum' Hamiltonian for these stochastic models. The simplest example is the problem of asymmetric diffusion of hard-core particles on the one-dimensional lattice (see [14–17] for reviews). The time evolution operator governing the time fluctuations of this last model coincides with the exact integrable anisotropic Heisenberg chain, or XXZ chain, in its ferromagnetic regime and appropriate boundary fields [17]. The related quantum chains in general are not exactly integrable but have their ground-state eigenfunctions given in terms of a matrix product ansatz. The matrix product ansatz, although providing only stationary properties of some stochastic system produced interesting results in quite a variety of problems including interface growth [18], boundary-induced phase transitions [13, 19–21], the dynamics of shocks [22] or traffic flow [23].

More recently an interesting development of the matrix product ansatz, named dynamical matrix product ansatz [24], emerged in the area of stochastic one-dimensional systems. Models satisfying this ansatz, distinct from the previous formulated matrix product ansatz, have their probability distributions, at arbitrary times, given in terms of a product of matrices, which are now time dependent. The dynamical matrix product ansatz was shown originally to be valid in the problem of asymmetric diffusion of particles on the lattice [24, 25]. More recently [26, 27] the validity of this ansatz was also confirmed in the exactly integrable manifold of the asymmetric diffusion of particles with two kinds of particles. The validity of this ansatz for such integrable system induces us to expect that all the quantum chains, related or not to the stochastic systems, that are solvable though the Bethe ansatz may also be solvable by an appropriate matrix product ansatz. We expect that all the components of an arbitrary eigenfunction of an exact integrable quantum chain, that are normally given in terms of a combination of plane waves, can also be expressed in terms of a product of matrices satisfying algebraic properties that ensure the exact integrability of the model.

In this paper, we are going to show the validity of this conjecture for a huge family of exactly integrable quantum Hamiltonians, by showing how to formulate their solutions in terms of a matrix product ansatz. A brief summary of some of our results has been announced in [28]. We are going to present explicit examples of a matrix product ansatz formulation for models having one or two global conservations. Examples of models with one global conservation law ($U(1)$ symmetric) include the XXZ quantum chain [29], the spin-1 Fateev–Zamolodchikov model [30], the Izergin–Korepin model [31] and the solvable spin-1 model considered in [32]. Among the models with two global conservation laws ($U(1) \times U(1)$ symmetric) we present the matrix product ansatz for the models where, in its formulation in terms of particles, no double occupancy of sites is allowed, such as the anisotropic spin-1 Sutherland model [33], the spin-1 Perk–Schultz models [34], the fermionic supersymmetric t–J model [35], and also for the models with double occupancy such as the spin-3/2

Perk–Schultz model [34], the Essler–Korepin–Schoutens model [36], the Hubbard model [37] and the two-parameter integrable model introduced in [38].

The ansatz we produce enables us to formulate in a simple and unified way generalized integrable models where the quantum spins have now hard-core interactions that exclude the occupation of two spins at neighbouring sites. Although most of our results will be presented for periodic chains they can also be extended for non-periodic but integrable chains. As an example of a non-periodic chain we present the solution, in terms of a matrix product ansatz, for the exact integrable XXZ chain with boundary z-magnetic fields [39].

The layout of the paper is as follows. In sections 2 and 3, we consider models with a single global conservation law. In section 2, we introduce and solve through our matrix product ansatz a generalization of the standard XXZ chain, where the up spins have a hard-core interaction of a given, but arbitrary, range in terms of units of lattice spacing. In section 3, models with spin 1 are considered. The matrix product ansatz is formulated for the spin-1 Fateev–Zamolodchikov model, the Izergin–Korepin model, as well as for the spin-1 model introduced in [32]. All these solutions are obtained through a single matrix product ansatz, whose matrices satisfy for each model distinct algebraic relations. In sections 4 and 5, we consider models with $U(1) \times U(1)$ symmetry, thus exhibiting two conserved global quantities. In section 4, we consider the models of spin 1 in this class. Those models are the anisotropic Sutherland model, or the $SU(3)$ Perk–Schultz model and the supersymmetric t–J model. In section 5, we present a general solution that includes the spin-3/2 Perk–Schultz model, the Essler–Korepin–Schoutens model, the Hubbard model as well as the general exact integrable two-parameter model presented in [38]. Again the solutions of all these models are presented through an unified matrix product ansatz. All the solutions presented in the previous sections are derived for quantum chains defined on lattices with toroidal boundary conditions; in section 6, we show how to extend our solutions for the case of the XXZ chain in an open lattice with a z-magnetic field at the end points of the lattice. Finally in section 7, we close our paper with some conclusions and comments.

2. Generalized XXZ chains

As a first application of our matrix product ansatz we are going to present, in this section, the exact solution of the anisotropic Heisenberg model or XXZ chain. In order to show the powerfulness of our ansatz, instead of solving the standard XXZ chain we are going to solve a generalization of this quantum Hamiltonian where arbitrary excluded volumes are considered among the up spins (σ^z-basis). We consider a generalized XXZ chain where any two up spins, due to hard-core interactions, are not allowed to occupy lattice sites at distances smaller than s ($s = 1, 2, \ldots$), in units of lattice spacing. Unlike the down spins, that have only on-site hard-core exclusions, the up spins behave as if they had an effective size s. The generalized XXZ Hamiltonian we consider, in a lattice with L sites, is given by

$$H_s = -\mathcal{P}_s \sum_{i=1}^{L} \frac{1}{2}\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+s}^z\right)\mathcal{P}_s \tag{1}$$

where σ^x, σ^y, σ^z are spin-1/2 Pauli matrices, Δ the anisotropy and the projector $\mathcal{P}_s$ projects out from the associated Hilbert space the configurations where any two up spins are at distances smaller than s. In the particular case where $s = 1$ the projector $\mathcal{P}_s$ is the unity operator and

we recover the standard XXZ chain. The Hamiltonian (1), in terms of raising and lowering spin-1/2 operators $\sigma^{\pm} = (\sigma^x \pm \mathrm{i}\sigma^y)/2$ is a particular case of the general Hamiltonian

$$H_s = -\mathcal{P}_s \sum_{i=1}^{L} \left[\epsilon_+ \sigma_i^- \sigma_{i+1}^+ + \epsilon_- \sigma_i^+ \sigma_{i+1}^- + \frac{\Delta}{2}\left(\sigma_i^z \sigma_{i+s}^z - 1\right) \right] \mathcal{P}_s \tag{2}$$

where $\epsilon_+ = \epsilon_- = 1$ and a harmless constant $(+L\Delta/2)$ was added. Without any additional difficulty we are going to consider the solution of (2), for general values of ϵ_+, ϵ_- and Δ. If we interpret the up spins as particles and the down spins as vacant sites the Hamiltonian (2) with the choice $\epsilon_+ + \epsilon_- = 1 = -2\Delta$ coincides, apart from a harmless constant, with the time evolution operator of the asymmetric diffusion problem (asymmetry ϵ_+/ϵ_-) of particles with size s on the lattice [40]. We are going to consider (2) with the periodic boundary condition

$$\sigma_{L+1}^{\pm} = \sigma_1^{\pm} \qquad \sigma_{L+1}^z = \sigma_1^z. \tag{3}$$

In section 6, we are going to consider the case of open boundary conditions.

The Hamiltonian (2) with the boundary condition (3) has a $U(1) \times Z_L$ symmetry due to its commutation with the total spin operator $S^z = \sum_{i=1}^{L} \sigma^z$ and the spatial translation operator $\hat{T} = \mathrm{e}^{\mathrm{i}\hat{P}}$ on the lattice. Consequently the Hilbert space associated with (2) can be separated into block disjoint sectors labelled by the number n of up spins ($n = 0, 1, \ldots, L$) and the momentum eigenvalues P $\left(P = \frac{2\pi}{L} l, l = 0, 1, \ldots, L-1\right)$.

The ansatz we propose asserts that any eigenfunction $|\Psi_{n,P}\rangle$ of (2) in the sector with n up spins and momentum P, will have its components given by the trace of the matrix product

$$\begin{aligned} |\Psi_{n,P}\rangle &= \sum_{\{x_1,\ldots,x_n\}}^{*} f(x_1, \ldots, x_n) |x_1, \ldots, x_n\rangle \\ f(x_1, \ldots, x_n) &= \mathrm{Tr}[E^{x_1-1} A^{(s)} E^{x_2-x_1-1} A^{(s)} \cdots E^{x_n-x_{n-1}-1} A^{(s)} E^{L-x_n} \Omega_P]. \end{aligned} \tag{4}$$

The configuration in σ^z-basis where the up spins are located at $(x_1, \ldots, x_n)$ are denoted by the ket $|x_1, \ldots, x_n\rangle$ and the symbol $(*)$ in the sum denotes the restriction to the sets satisfying the hard-core exclusion due to the effective size s of the up spins, i.e.,

$$x_{i+1} \geqslant x_i + s \qquad i = 1, \ldots, n-1 \qquad x_1 \geqslant 1 \qquad s \leqslant x_n - x_1 \leqslant L - s. \tag{5}$$

Differently from the standard Bethe ansatz where $f(x_1, \ldots, x_n)$ is given by the combination of plane waves now it is given by the trace of a product of matrices. Actually E, $A^{(s)}$ and Ω_P are abstract objects with appropriate commutation relations. These operators in order to provide well-defined amplitudes for the eigenfunctions should have an associative product, and as usual in the literature, we call them simply matrices. The matrices E and $A^{(s)}$ are associated with the down and up spins representing the ket configuration. The superscript s is just to remember that the up spins have an effective size s. The matrix Ω_P in (4) is introduced in order to fix the momentum P of the eigenfunction $|\Psi_{n,P}\rangle$. This is accomplished by imposing the algebraic relations[1]

$$E\Omega_P = \mathrm{e}^{-\mathrm{i}P} \Omega_P E \qquad A^{(s)} \Omega_P = \mathrm{e}^{-\mathrm{i}P} \Omega_P A^{(s)} \tag{6}$$

since from (4), eigenfunctions of momentum P should have the ratio of amplitudes

$$\frac{f(x_1, \ldots, x_n)}{f(x_1 + m, \ldots, x_n + m)} = \mathrm{e}^{-\mathrm{i}Pm} \qquad (m = 0, 1, \ldots, L-1). \tag{7}$$

[1] In the general case of twisted boundary conditions $\sigma_{L+1}^{\pm} = \mathrm{e}^{\pm \mathrm{i}\Phi} \sigma_1^{\pm}$, $\sigma_{L+1}^z = \sigma_1^z$ the ansatz also works with an appropriate generalization of (6).

The matrix product ansatz for $|\Psi_{n,P}\rangle$ will work if we are able to obtain consistent algebraic relations among E and $A^{(s)}$ that solve the eigenvalue equation

$$H_s|\Psi_{n,P}\rangle = \varepsilon_n|\Psi_{n,P}\rangle. \tag{8}$$

As is customary, before considering the case of general values of n let us consider the eigensectors with $n = 1$ and $n = 2$ up spins.

$n = 1$. For one up spin the eigenvalue equation (8) gives us for the amplitudes with a single spin at $x_1 = 1, \ldots, L$

$$\begin{aligned}\varepsilon_1 \mathrm{Tr}[E^{x_1-1}A^{(s)}E^{L-x_1}\Omega_P] = &-\epsilon_+\mathrm{Tr}[E^{x_1-2}A^{(s)}E^{L-x_1+1}\Omega_P] \\ &-\epsilon_-\mathrm{Tr}[E^{x_1}A^{(s)}E^{L-x_1-1}\Omega_P] + \Delta\mathrm{Tr}[E^{x_1-1}A^{(s)}E^{L-x_1}\Omega_P].\end{aligned} \tag{9}$$

The cyclic property of the trace and relation (6) fix the eigenenergies

$$\varepsilon_1 = -\left(\epsilon_+ \mathrm{e}^{-\mathrm{i}P} + \epsilon_- \mathrm{e}^{\mathrm{i}P} - \Delta\right) \tag{10}$$

where $P = \frac{2\pi l}{L}$ $(l = 0, 1, \ldots, L-1)$ is the momentum of the state. An alternative solution of (9) that will be simple to generalize for arbitrary values of n is obtained by replacing

$$A^{(s)} = A_k^{(s)}E^{2-s} \tag{11}$$

where now $A_k^{(s)}$ is a spectral parameter dependent matrix obeying the following commutation relation with the matrix E:

$$EA_k^{(s)} = \mathrm{e}^{\mathrm{i}k}A_k^{(s)}E. \tag{12}$$

Inserting (11) in (9) and using (12) we obtain

$$\varepsilon_1 = \varepsilon(k) = -(\epsilon_+ \mathrm{e}^{-\mathrm{i}k} + \epsilon_- \mathrm{e}^{\mathrm{i}k} - \Delta) \tag{13}$$

and comparing (10) with (13) we fix the spectral parameter k as the momentum of the 1-particle eigenfunction $|\Psi_{1,P}\rangle$, i.e., $k = P = \frac{2\pi l}{L}$ $(l = 0, 1, \ldots, L-1)$.

$n = 2$. The eigenvalue equation (8) gives for the amplitudes of $|\Psi_{2,P}\rangle$ two types of relations depending on whether the two up spins located at x_1 and x_2 are at the closest position or not. The amplitudes corresponding to the kets $|x_1, x_2\rangle$ where $x_2 > x_1 + s$ give us the relation

$$\begin{aligned}\varepsilon_2 \mathrm{Tr}[E^{x_1-1}A^{(s)}E^{x_2-x_1-1}A^{(s)}E^{L-x_2}\Omega_P] = &-\epsilon_+\mathrm{Tr}[E^{x_1-2}A^{(s)}E^{x_2-x_1}A^{(s)}E^{L-x_2}\Omega_P] \\ &-\epsilon_-\mathrm{Tr}[E^{x_1}A^{(s)}E^{x_2-x_1-2}A^{(s)}E^{L-x_2}\Omega_P] \\ &-\epsilon_+\mathrm{Tr}[E^{x_1-1}A^{(s)}E^{x_2-x_1-2}A^{(s)}E^{L-x_2+1}\Omega_P] \\ &-\epsilon_-\mathrm{Tr}[E^{x_1-1}A^{(s)}E^{x_2-x_1}A^{(s)}E^{L-x_2-1}\Omega_P] \\ &+2\Delta\mathrm{Tr}[E^{x_1-1}A^{(s)}E^{x_2-x_1-1}A^{(s)}E^{L-x_2}\Omega_P].\end{aligned} \tag{14}$$

A possible and convenient way to solve this equation is obtained by a generalization of (11) and (12). We identify $A^{(s)}$ as composed by two spectral parameter dependent new matrices $A_{k_1}^{(s)}$ and $A_{k_2}^{(s)}$, i.e.,

$$A^{(s)} = \sum_{i=1}^{2} A_{k_i}^{(s)}E^{2-s} \tag{15}$$

that satisfy the commutation relations

$$E A_{k_j}^{(s)} = \mathrm{e}^{\mathrm{i}k_j} A_{k_j}^{(s)} E \qquad \left(A_{k_j}^{(s)}\right)^2 = 0 \qquad j = 1, 2. \tag{16}$$

Inserting (15) in (14) and using (16) we obtain

$$\varepsilon_2 = \varepsilon(k_1) + \varepsilon(k_2) \tag{17}$$

where $\varepsilon(k)$ is given by (13). The relation (6) gives us the commutation of the new matrices $A_{k_i}^{(s)}$ with Ω_P, i.e.,

$$A_{k_j}^{(s)} \Omega_P = \mathrm{e}^{\mathrm{i}P(1-s)} \Omega_P A_{k_j}^{(s)} \qquad j = 1, 2. \tag{18}$$

Comparing as in (7) the amplitudes of the configurations $|x_1, x_2\rangle$ and $|x_1 + m, x_2 + m\rangle$ and exploring the cyclic property of the trace we obtain

$$P = k_1 + k_2. \tag{19}$$

Up to now we have at our disposal, for solving the eigenvalue equation, the commutation relation of $A_{k_1}^{(s)}$ and $A_{k_2}^{(s)}$ as well as the spectral parameters k_1 and k_2, that might be in general complex numbers. These commutation relations are going to be fixed by the application of the eigenvalue equation (8) to the components of the configurations $|x_1, x_2\rangle$ where the up spins are at the 'matching' condition $x_2 = x_1 + s$:

$$\begin{aligned}\varepsilon_2 \mathrm{Tr}[E^{x_1-1} A^{(s)} E^{s-1} A^{(s)} E^{L-x_1-s} \Omega_P] &= -\epsilon_+ \mathrm{Tr}[E^{x_1-2} A^{(s)} E^{s} A^{(s)} E^{L-x_1-s} \Omega_P] \\ &\quad - \epsilon_- \mathrm{Tr}[E^{x_1-1} A^{(s)} E^{s} A^{(s)} E^{L-x_1-s-1} \Omega_P] \\ &\quad + \Delta \mathrm{Tr}[E^{x_1-1} A^{(s)} E^{s-1} A^{(s)} E^{L-x_1-s} \Omega_P].\end{aligned} \tag{20}$$

Using (15), (17) and the commutation relations (16) in the last expression one obtains

$$\sum_{j,l=1}^{2} \left[\epsilon_- + \epsilon_+ \mathrm{e}^{-\mathrm{i}(k_j+k_l)} - \Delta \mathrm{e}^{-\mathrm{i}k_j}\right] A_{k_j}^{(s)} A_{k_l}^{(s)} = 0. \tag{21}$$

This implies that the matrices $A_{k_j}^{(s)}$ $(j = 1, 2)$ should obey

$$A_{k_j}^{(s)} A_{k_l}^{(s)} = S(k_j, k_l) A_{k_l}^{(s)} A_{k_j}^{(s)} \qquad (l \neq j) \qquad \left(A_{k_j}^{(s)}\right)^2 = 0 \qquad (j, l = 1, 2) \tag{22}$$

where

$$S(k_j, k_l) = -\frac{\epsilon_+ + \epsilon_- \mathrm{e}^{\mathrm{i}(k_j+k_l)} - \Delta \mathrm{e}^{\mathrm{i}k_j}}{\epsilon_+ + \epsilon_- \mathrm{e}^{\mathrm{i}(k_j+k_l)} - \Delta \mathrm{e}^{\mathrm{i}k_l}}. \tag{23}$$

This last relation in the context of (1 + 1)-dimensional field theory is known as the Zamolodchikov algebra [41, 42]. The complex spectral parameters k_j $(j = 1, 2)$ that are still free up to now are fixed by imposing that the ratio of two components $f(x_1, x_2)/f(x_1', x_2')$ should be uniquely related. The cyclic property of the trace with the algebraic relations (16), (17) and (22) give us

$$\begin{aligned}\mathrm{Tr}\left[A_{k_l}^{(s)} A_{k_j}^{(s)} E^{L-2s+2} \Omega_P\right] &= \mathrm{e}^{-\mathrm{i}(L-2s+2)k_j} \mathrm{Tr}\left[A_{k_l}^{(s)} E^{L-2s+2} A_{k_j}^{(s)} \Omega_P\right] \\ &= \mathrm{e}^{-\mathrm{i}k_j L} \mathrm{e}^{\mathrm{i}2k_j(s-1)} \mathrm{e}^{-\mathrm{i}P(s-1)} S(k_j, k_l) \mathrm{Tr}\left[A_{k_l}^{(s)} A_{k_j}^{(s)} E^{L-2s+2} \Omega_P\right]\end{aligned} \tag{24}$$

or equivalently, since $P = k_1 + k_2$,

$$\mathrm{e}^{\mathrm{i}k_j L} = S(k_j, k_l) \left(\frac{\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}k_l}}\right)^{s-1} \qquad j = 1, 2 \quad (j \neq l). \tag{25}$$

The energies ε_2 and momentum P of $|\Psi_{2,P}\rangle$ are obtained by inserting the solutions of (25) into (17) and (19) respectively.

General n. The previous calculation can easily be extended for arbitrary values of the number n of up spins. The eigenvalue equation (8) when applied to the amplitudes of $|\Psi_{n,P}\rangle$ corresponding to the configurations where all the n spins are at distances larger than the excluded volume s, gives a generalization of (14):

$$\begin{aligned}\varepsilon_n \mathrm{Tr}[\cdots E^{x_i-x_{i-1}-1}A^{(s)}E^{x_{i+1}-x_i-1}A^{(s)}\cdots A^{(s)}E^{L-x_n}\Omega_P] \\ = -\sum_{i=1}^{n}\{\epsilon_+\mathrm{Tr}[\cdots E^{x_i-x_{i-1}-2}A^{(s)}E^{x_{i+1}-x_i}A^{(s)}\cdots A^{(s)}E^{L-x_n}\Omega_P] \\ +\epsilon_-\mathrm{Tr}[\cdots E^{x_i-x_{i-1}-1}A^{(s)}E^{x_{i+1}-x_i-2}A^{(s)}\cdots A^{(s)}E^{L-x_n+1}\Omega_P] \\ -\Delta\mathrm{Tr}[\cdots E^{x_i-x_{i-1}-1}A^{(s)}E^{x_{i+i}-x_i-1}A^{(s)}\cdots A^{(s)}E^{L-x_n}\Omega_P]\}. \end{aligned} \tag{26}$$

The solution of this last equation[2] is obtained by identifying the $A^{(s)}$ matrix as a combination of n spectral parameter dependent matrices $\{A_{k_j};\ j=1,\ldots,n\}$, i.e.,

$$A^{(s)} = \sum_{j=1}^{n} A_{k_j}^{(s)} E^{2-s} \tag{27}$$

with the commutation relations with the matrices E and Ω_P given by

$$EA_{k_j}^{(s)} = \mathrm{e}^{\mathrm{i}k_j}A_{k_j}^{(s)}E \qquad A_{k_j}^{(s)}\Omega_P = \mathrm{e}^{\mathrm{i}P(1-s)}\Omega_P A_{k_j}^{(s)} \qquad (j=1,\ldots,n). \tag{28}$$

The energy and momentum are obtained by inserting (27) into (26) and using (28), and are given by

$$\varepsilon_n = \sum_{j=1}^{n}\varepsilon(k_j) \qquad P = \sum_{j=1}^{n} k_j \tag{29}$$

respectively. The eigenvalue equation (8) gives for the configuration where a pair of up spins are at the 'colliding' positions $x_{i+1} = x_i + s$ a relation that coincides with (21)–(23) but with $j, l = 1,\ldots,n$. The relation coming from the configuration where three particles are at the colliding position $x_{i+1} = x_i + s$, $x_{i+2} = x_{i+1} + s$ is given by

$$\begin{aligned}\sum_{j,l=1}^{n}\left(\epsilon_- + \epsilon_+\mathrm{e}^{-\mathrm{i}(k_j+k_l)} - \Delta\,\mathrm{e}^{-\mathrm{i}k_j}\right)A_{k_j}^{(s)}A_{k_l}^{(s)}\sum_{t=1}^{n}A_{k_t}^{(s)}\mathrm{e}^{\mathrm{i}k_t} \\ +\sum_{j=1}^{n}\mathrm{e}^{-\mathrm{i}k_j}A_{k_j}^{(s)}\sum_{j,l=1}^{n}\mathrm{e}^{\mathrm{i}(k_l+k_t)}\left(\epsilon_- + \epsilon_+\mathrm{e}^{-\mathrm{i}(k_l+k_t)} - \Delta\,\mathrm{e}^{-\mathrm{i}k_l}\right)A_{k_l}^{(s)}A_{k_t}^{(s)} = 0 \end{aligned} \tag{30}$$

being a consequence of the relation (21). Similarly the amplitudes with arbitrary number of particles at colliding position will be automatically satisfied if the matrices $E, A_{k_j}^{(s)}$ $(j=1,\ldots,n)$ and Ω_P obey the algebraic relations (22) and (28). The associativity of the algebra provide a well-defined value for any product of matrices and follows from the fact that the structure constants $S(k_i,k_j)$ in (22) and (23) are c-numbers with the property $S(k_i,k_j)S(k_j,k_i) = 1$ $(i,j=1,\ldots,n)$.

The cyclic invariance of the trace in (4) will fix the complex spectral parameters k_j $(j=1,\ldots,n)$, providing a well-defined value for the components of $|\Psi_{n,P}\rangle$, i.e.,

[2] The most general relation $A^{(s)} = \sum_{j=1}^{n} E^{\alpha}A_{k_j}^{(s)}E^{\beta}$ could be used. However (27) is more convenient since otherwise the structure constants $S(k_i,k_j)$ in (23) will depend on the size s.

$$\mathrm{Tr}\left[A_{k_1}^{(s)}\cdots A_{k_j}^{(s)}\cdots A_{k_n}^{(s)}E^{L-n(s-1)}\Omega_P\right] = \mathrm{e}^{-\mathrm{i}k_j[L-n(s-1)]}\,\mathrm{e}^{\mathrm{i}P(1-s)} \times\prod_{l=j+1}^{n} S(k_j,k_l)\,\mathrm{Tr}\left[A_{k_j}^{(s)}A_{k_1}^{(s)}\cdots A_{k_{j-1}}^{(s)}A_{k_{j+1}}^{(s)}\cdots A_{k_n}^{(s)}E^{L-n(s-1)}\Omega_P\right]$$

$$= \mathrm{e}^{-\mathrm{i}k_j[L-n(s-1)]}\,\mathrm{e}^{\mathrm{i}P(1-s)}\prod_{l=1,l\neq j}^{n} S(k_j,k_l)\,\mathrm{Tr}\left[A_{k_1}^{(s)}\cdots A_{k_j}^{(s)}\cdots A_{k_n}^{(s)}E^{L-n(s-1)}\Omega_P\right] \qquad j=1,\ldots,n \tag{31}$$

or equivalently

$$\mathrm{e}^{\mathrm{i}k_jL} = (-1)^n\prod_{l=1}^{n}\left(\frac{\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}k_l}}\right)^{s-1}\frac{\epsilon_+ + \epsilon_-\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} - \Delta\,\mathrm{e}^{\mathrm{i}k_j}}{\epsilon_+ + \epsilon_-\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} - \Delta\,\mathrm{e}^{\mathrm{i}k_l}} \qquad j=1,\ldots,n. \tag{32}$$

The acceptable set $\{k_j;\, j=1,\ldots,n\}$ of spectral parameters defining the eigenvectors $|\Psi_{n,P}\rangle$ are the solutions of (32) where $k_i \neq k_j$ $(i,j=1,\ldots,n)$. Since $\left(A_k^{(s)}\right)^2 = 0$, solutions of (32) with coinciding roots give us null states.

Equation (32) coincides with the Bethe ansatz equation derived for the model (2) through the standard Bethe ansatz [40]. The choice $\epsilon_+ = \epsilon_- = 1$ with $s=1$ gives the Bethe ansatz equation of the standard XXZ chain (1) [29]. Moreover, using (27), an arbitrary component $f(x_1,\ldots,x_n)$ of the eigenfunction $|\Psi_{n,P}\rangle$, given by the ansatz (4), can be written as

$$f(x_1,\ldots,x_n) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} \times\mathrm{Tr}\left[E^{x_1-1}A_{k_{i_1}}^{(s)}E^{x_2-x_1+1-s}A_{k_{i_2}}^{(s)}\cdots E^{x_n-x_{n-1}+1-s}A_{k_{i_n}}^{(s)}E^{L-x_n+2-s}\Omega_P\right]. \tag{33}$$

The commutation relations (28) allow us to write

$$f(x_1,\ldots,x_n) = \sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}\mathrm{e}^{\mathrm{i}[k_{i_1}(x_1-1)+\cdots+k_{i_n}(x_n-1)]} \times\mathrm{Tr}\left[A_{k_{i_1}}^{(s)}E^{1-s}A_{k_{i_2}}^{(s)}E^{1-s}\cdots E^{1-s}A_{k_{i_n}}^{(s)}E^{L}\Omega_P\right]. \tag{34}$$

Let us define the new matrices

$$\tilde{A}_{k_j}^{(s)} = A_{k_j}^{(s)}E^{1-s} \qquad (j=1,\ldots,n). \tag{35}$$

It is simple to verify, from (22), that they satisfy, for $j,l=1,\ldots,n$

$$\tilde{A}_{k_j}^{(s)}\tilde{A}_{k_l}^{(s)} = \tilde{S}(k_j,k_l)\tilde{A}_{k_l}^{(s)}\tilde{A}_{k_j}^{(s)} \qquad (j\neq l) \qquad \left(\tilde{A}_{k_j}^{(s)}\right)^2 = 0 \tag{36}$$

where

$$\tilde{S}(k_j,k_l) = S(k_j,k_l)\left(\frac{\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}k_l}}\right)^{s-1}. \tag{37}$$

In terms of these new matrices, and exploring the fact that $\left(\tilde{A}_{k_j}^{(s)}\right)^2 = 0$ we can write

$$f(x_1,\ldots,x_n) = \sum_{p_1,\ldots,p_n}\mathrm{e}^{\mathrm{i}[k_{p_1}(x_1-1)+\cdots+k_{p_n}(x_n-1)]}\,\mathrm{Tr}\left[\tilde{A}_{k_{p_1}}^{(s)}\cdots\tilde{A}_{k_{p_n}}^{(s)}E^{L}\Omega_P\right] \tag{38}$$

where the sum is over the permutations $\{p_1,p_2,\ldots,p_n\}$ of the non-repeated integers $(1,\ldots,n)$. This last result showns us that the amplitudes derived using the proposed matrix product ansatz (4) is given by a combination of plane waves with complex wavenumbers $\{k_i\}$, and reproduces the results previously obtained for the Hamiltonian (2) [40] through the coordinate Bethe ansatz.

3. Models of spin 1 with one conservation law

We present in this section the appropriate matrix ansatz solution for models of spin 1 with a single global conservation law, such as the XXZ chain of the previous section. Integrable models in this category are the Fateev–Zamolodchikov quantum chain [30], the Izergin–Korepin model [31] and the spin-1 model introduced in [32].

We start with a general spin-1 model with $U(1)$ symmetry and nearest-neighbour interaction. Instead of writing this general model in terms of spin-1 Pauli matrices it is more convenient to write it in terms of the 3×3 Weyl matrices $E^{l,m}$ $(l, m = 0, 1, 2)$, with i, j elements $(E^{l,m})_{i,j} = \delta_{l,j}\delta_{m,j}$. At each lattice site we may have zero particles ($s_z = -1$), one particle ($s_z = 0$) or two particles ($s_z = 1$). The general Hamiltonian we consider, that conserves the number of particles, is given by

$$\begin{aligned} H^{U(1)}_{s=1} &= -\sum_{i=1}^{L} h_{i,i+1} + L\Gamma^{0\,0}_{0\,0}, \\ h_{i,i+1} &= \sum_{k,l,m,o=0}^{2} \Gamma^{kl}_{mo} E^{m,k}_i E^{o,l}_{i+1} \qquad (i = 1, \ldots, L) \end{aligned} \tag{39}$$

where periodic boundary conditions are imposed and in order to conserve the total number of particles $\Gamma^{kl}_{mo} = 0$ if $k + l \neq m + o$.

The total number of particles n $(0, 1, \ldots, 2L)$ and the momentum P $\left(\frac{2\pi l}{L}, l = 0, \ldots, L - 1\right)$ are good quantum numbers for the general Hamiltonian (39). The eigenfunctions on these eigensectors are given by

$$|\Psi_{n,P}\rangle = \sum_{\{x_1,\ldots,x_n\}}^{*} f(x_1, \ldots, x_n)|x_1, \ldots, x_n\rangle \tag{40}$$

where $(x_1, \ldots, x_n)$ are the coordinates of the particles and the symbol $(*)$ in the sum means restriction to the sets with coordinates $\{x_{i+1} \geqslant x_i, x_{i+2} > x_i\}$.

In order to formulate an appropriate matrix product ansatz for (39) we associate, as in the previous section, the matrices E and A with the empty sites ($s_z = -1$) and the sites occupied by a single particle ($s_z = 0$), respectively. The sites with double occupancy of particles ($s_z = 1$) are associated with the matrix $BE^{-1}B$. Certainly the Hamiltonian (39) is not exactly integrable for arbitrary values of $\left\{\Gamma^{kl}_{mo}\right\}$. The ansatz we propose states that in the exact integrable manifold of (39) any eigenfunction $|\Psi_{n,P}\rangle$, in the sector with n particles and momentum P will have components given in terms of traces of a matrix product. The amplitudes corresponding to the configurations where there is no double occupancy are given by

$$f(x_1, \ldots, x_n) = \mathrm{Tr}\,[E^{x_1-1}AE^{x_2-x_1-1}A \cdots E^{x_n-x_{n-1}-1}AE^{L-x_n}\Omega_P] \tag{41}$$

while if there exists a double occupancy at $x_{i+1} = x_i$ we have

$$\begin{aligned} &f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \\ &\qquad = \mathrm{Tr}[E^{x_1-1}A \cdots E^{x_i-x_{i-1}-1}BE^{-1}BE^{x_{i+2}-x_i-1} \cdots E^{x_n-x_{n-1}-1}AE^{L-x_n}\Omega_P]. \end{aligned} \tag{42}$$

As in the previous section the matrix Ω_P and the trace operation are introduced in order to fix the momentum $P = \frac{2\pi}{L}l$ $(l = 0, 1, \ldots, L - 1)$ of $|\Psi_{n,P}\rangle$, this is accomplished (see (7)) by imposing

$$E\Omega_P = \mathrm{e}^{-\mathrm{i}P}\Omega_P E \qquad A\Omega_P = \mathrm{e}^{-\mathrm{i}P}\Omega_P A \qquad B\Omega_P = \mathrm{e}^{-\mathrm{i}P}\Omega_P B. \tag{43}$$

The algebraic relations among the E, A and B operators will be obtained from the requirement that (39) with the ansatz (40)–(42) are solutions of the eigenvalue equation

$$H_{s=1}^{U(1)}|\Psi_{n,P}\rangle = \varepsilon_n|\Psi_{n,P}\rangle. \tag{44}$$

Let us consider initially the cases with a small number of particles.

$n = 1$. The eigenvalue equation gives for the amplitudes with a particle at position $x_1 = 1, \ldots, L$:

$$\begin{aligned}\varepsilon_1 \mathrm{Tr}[E^{x_1-1}AE^{L-x_1}\Omega_P] &= -\Gamma^{1\,0}_{0\,1}\,\mathrm{Tr}[E^{x_1-2}AE^{L-x_1+1}\Omega_P] - \Gamma^{0\,1}_{1\,0}\,\mathrm{Tr}[E^{x_1}AE^{L-x_1-1}\Omega_P] \\ &\quad + \left(2\Gamma^{0\,0}_{0\,0} - \Gamma^{1\,0}_{1\,0} - \Gamma^{0\,1}_{0\,1}\right)\mathrm{Tr}[E^{x_1-1}AE^{L-x_1}\Omega_P].\end{aligned} \tag{45}$$

As in the previous section, a convenient solution is obtained by introducing the spectral parameter matrix A_k,

$$A = A_k E \qquad \text{with} \qquad EA_k = \mathrm{e}^{\mathrm{i}k}A_k E. \tag{46}$$

Inserting (46) into (44) we obtain

$$\begin{aligned}&\varepsilon_1 = \varepsilon(k) = -\left(\Gamma^{1\,0}_{0\,1}\,\mathrm{e}^{-\mathrm{i}k} + \Gamma^{0\,1}_{1\,0}\,\mathrm{e}^{\mathrm{i}k} - 2\Gamma^{0\,0}_{0\,0} + \Gamma^{1\,0}_{1\,0} + \Gamma^{0\,1}_{0\,1}\right) \\ &P = k = \frac{2\pi l}{L} \qquad (l = 0, \ldots, L-1).\end{aligned} \tag{47}$$

$n = 2$. In this case the eigenvalue equation (43) produces relations that are distinct if $x_2 > x_1 + 1$, $x_2 = x_1 + 1$ or $x_2 = x_1$. The relations coming from the amplitudes where the particles are not at the 'colliding' positions $x_2 = x_1$ or $x_2 = x_1 + 1$ are just a straightforward generalization of (45), for two particles, whose solution is obtained by identifying the matrix A as composed by two spectral parameter dependent matrices A_{k_1}, A_{k_2}, i.e.,

$$A = \sum_{j=1}^{n} A_{k_j} E \tag{48}$$

satisfying the commutations relations

$$EA_{k_j} = \mathrm{e}^{\mathrm{i}k_j}A_{k_j}E \qquad \left(A_{k_j}\right)^2 = 0 \qquad j = 1, \ldots, n. \tag{49}$$

The energy and momentum are given by

$$\varepsilon_n = \sum_{j=1}^{n} \varepsilon(k_j)P = \sum_{j=1}^{n} k_j \tag{50}$$

with $\varepsilon(k)$ given by (47). The eigenvalue equation (44) gives for the components $f(x_1, x_1+1)$ the equation

$$\begin{aligned}\varepsilon_2 \mathrm{Tr}[E^{x_1-1}AAE^{L-x_1-1}\Omega_P] &= -\Gamma^{1\,0}_{0\,1}\,\mathrm{Tr}[E^{x_1-2}AEAE^{L-x_1-1}\Omega_P] \\ &\quad - \Gamma^{0\,1}_{1\,0}\,\mathrm{Tr}[E^{x_1-1}AEAE^{L-x_1-2}\Omega_P] - \Gamma^{2\,0}_{1\,1}\,\mathrm{Tr}[E^{x_1-1}BE^{-1}BE^{L-x_1}\Omega_P] \\ &\quad - \Gamma^{0\,2}_{1\,1}\,\mathrm{Tr}[E^{x_1}BE^{-1}BE^{L-x_1-1}\Omega_P] + \left(3\Gamma^{0\,0}_{0\,0} - \Gamma^{1\,0}_{1\,0} - \Gamma^{0\,1}_{0\,1} - \Gamma^{1\,1}_{1\,1}\right) \\ &\quad \times \mathrm{Tr}[E^{x_1-1}AAE^{L-x_1-1}\Omega_P]\end{aligned} \tag{51}$$

while the components $f(x_1, x_1)$ give

$$\begin{aligned}\varepsilon_2 \mathrm{Tr}[E^{x_1-1}BE^{-1}BE^{L-x_1}\Omega_P] &= -\Gamma^{1\,1}_{0\,2}\,\mathrm{Tr}[E^{x_1-2}AAE^{L-x_1}\Omega_P] \\ &\quad - \Gamma^{1\,1}_{2\,0}\,\mathrm{Tr}[E^{x_1-1}AAE^{L-x_1-1}\Omega_P] - \Gamma^{2\,0}_{0\,2}\,\mathrm{Tr}[E^{x_1-2}BE^{-1}BE^{L-x_1+1}\Omega_P] \\ &\quad - \Gamma^{0\,2}_{2\,0}\,\mathrm{Tr}[E^{x_1}BE^{-1}BE^{L-x_1-1}\Omega_P] \\ &\quad + \left(2\Gamma^{0\,0}_{0\,0} - \Gamma^{2\,0}_{2\,0} - \Gamma^{0\,2}_{0\,2}\right)\mathrm{Tr}[E^{x_1-1}BE^{-1}BE^{L-x_1}\Omega_P].\end{aligned} \tag{52}$$

Both equations (51), (52) are solved by using (48) and by writing the matrix B also as a combination of two spectral parameter dependent matrices B_{k_1}, B_{k_2}, i.e.,

$$B = \sum_{j=1}^{n} B_{k_j} E \qquad \text{with} \quad E B_{k_j} = \mathrm{e}^{\mathrm{i}k_j} B_{k_j} E \qquad \left(B_{k_j}\right)^2 = 0 \qquad (j = 1, \ldots, n). \tag{53}$$

Note that the matrices A_{k_j}, B_{k_j} have the same set $\{k_1, k_2\}$ of undetermined complex spectral parameters. Inserting (48)–(50) and (53) in (51) and (52) we obtain

$$\sum_{j,l=1}^{n} N(k_j, k_l)\,\mathrm{Tr}\left[E^{x_1} A_{k_j} A_{k_l} E^{L-x_1} \Omega_P\right] = \sum_{j,l=1}^{n} C_1(k_j, k_l)\,\mathrm{Tr}\left[E^{x_1} B_{k_j} B_{k_l} E^{L-x_1} \Omega_P\right] \tag{54}$$

and

$$\sum_{j,l=1}^{n} C_0(k_j, k_l)\,\mathrm{Tr}\left[E^{x_1} B_{k_j} B_{k_l} E^{L-x_1} \Omega_P\right] = \sum_{j,l=1}^{n} C_2(k_j, k_l)\,\mathrm{e}^{\mathrm{i}k_l}\,\mathrm{Tr}\left[E^{x_1} A_{k_j} A_{k_l} E^{L-x_1} \Omega_P\right] \tag{55}$$

where

$$\begin{aligned}
N(k_j, k_l) &= \Gamma^{1\,0}_{0\,1} + \left(\Gamma^{1\,0}_{1\,0} + \Gamma^{0\,1}_{0\,1} - \Gamma^{0\,0}_{0\,0} - \Gamma^{1\,1}_{1\,1}\right)\mathrm{e}^{\mathrm{i}k_l} + \Gamma^{0\,1}_{1\,0}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} \\
C_1(k_j, k_l) &= \Gamma^{2\,0}_{1\,1} + \Gamma^{0\,2}_{1\,1}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} \\
C_0(k_j, k_l) &= \Gamma^{1\,0}_{0\,1}(\mathrm{e}^{\mathrm{i}k_l} + \mathrm{e}^{\mathrm{i}k_j}) + \Gamma^{0\,1}_{1\,0}(\mathrm{e}^{\mathrm{i}k_l} + \mathrm{e}^{\mathrm{i}k_j})\,\mathrm{e}^{\mathrm{i}(k_l+k_j)} \\
&\quad + \left(2\Gamma^{1\,0}_{1\,0} + 2\Gamma^{0\,1}_{0\,1} - \Gamma^{2\,0}_{2\,0} - \Gamma^{0\,2}_{0\,2} - 2\Gamma^{0\,0}_{0\,0}\right)\mathrm{e}^{\mathrm{i}(k_j+k_l)} - \Gamma^{2\,0}_{0\,2} - \Gamma^{0\,2}_{2\,0}\,\mathrm{e}^{\mathrm{i}2(k_j+k_l)} \\
C_2(k_j, k_l) &= \Gamma^{1\,1}_{0\,2} + \Gamma^{1\,1}_{2\,0}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)}.
\end{aligned} \tag{56}$$

The relations (54) and (55) imply

$$N(k_j, k_l) A_{k_j} A_{k_l} + N(k_l, k_j) A_{k_l} A_{k_j} = C_1(k_j, k_l)\left(B_{k_j} B_{k_l} + B_{k_l} B_{k_j}\right) \tag{57}$$

$$C_0(k_j, k_l)\left(B_{k_j} B_{k_l} + B_{k_l} B_{k_j}\right) = C_2(k_j, k_l)\left(\mathrm{e}^{\mathrm{i}k_l} A_{k_j} A_{k_l} + \mathrm{e}^{\mathrm{i}k_j} A_{k_l} A_{k_j}\right) \tag{58}$$

with $k, l = 1, \ldots, n$. Multiplying (58) by the symmetric function $C_1(k_j, k_l)$ and using (57), the last equation gives the relation

$$A_{k_j} A_{k_l} = S(k_j, k_l) A_{k_l} A_{k_j} \qquad (j \neq l)$$

$$S(k_j, k_l) = -\frac{C_0(k_j, k_l) N(k_l, k_j) - C_1(k_j, k_l) C_2(k_j, k_l)\,\mathrm{e}^{\mathrm{i}k_j}}{C_0(k_j, k_l) N(k_j, k_l) - C_1(k_j, k_l) C_2(k_j, k_l)\,\mathrm{e}^{\mathrm{i}k_l}} \qquad j, l = 1, 2. \tag{59}$$

The spectral parameters k_1 and k_2 are going to be fixed by the cyclic property of the traces defining the amplitudes (41) and (42). Since the B matrices only appear in the combination

$$B E^{-1} B = \left(\sum_{j,l=1}^{n} B_{k_j} B_{k_l}\right) E. \tag{60}$$

The use of (57) allows us to replace in any amplitude the $\{B_k\}$ matrices by the $\{A_k\}$ matrices. Then an arbitrary amplitude should be proportional to $\mathrm{Tr}\left[A_{k_j} A_{k_j} E^L \Omega_P\right]$. The cyclic property of the trace and the commutation relations (43), (49) and (59) give us

$$\mathrm{e}^{\mathrm{i}k_j L} = S(k_j, k_l) \qquad j = 1, 2 \quad (j \neq l) \tag{61}$$

with $S(k_j, k_l)$ given by (59). It is interesting to mention that we obtained for $n = 2$ particles the integrability for arbitrary values of the coupling Γ^{kl}_{mo} of the Hamiltonian (39). Certainly this will be not the case for $n > 2$ particles.

$n = 3$. In this case we have several distinct types of relations for the amplitudes $f(x_1, x_2, x_3)$ in (41) and (42). The eigenvalue equation when applied to the amplitudes where all the

particles are not at 'colliding' positions, i.e., $x_3 > x_2 + 1 > x_1 + 2$ give us a straightforward generalization of (45), for three particles, whose solution is obtained by identifying, as in (48), the matrix A as composed by three spectral parameter dependent matrices A_{k_1}, A_{k_2}, A_{k_3}. The energy and momentum are given by (50) with $n = 3$. The components $f(x_1, x_2 = x_1, x_3)$, $f(x_1, x_2 = x_1 + 1, x_3)$ with $x_3 > x_2 + 1$ and $f(x_1, x_2, x_3 = x_2)$, $f(x_1, x_2, x_3 = x_2 + 1)$ with $x_1 < x_2 - 1$ give us generalizations of (57) and (58) for $n = 3$. We have new relations when the eigenvalue equation is applied to the amplitudes corresponding to the three particles at the colliding positions. This happens for $f(x, x, x + 1)$

$$\begin{aligned}\varepsilon_3 \mathrm{Tr}[E^{x-1}BE^{-1}BAE^{L-x-1}\Omega_P] &= -\Gamma^{1\,1}_{0\,2}\,\mathrm{Tr}[E^{x-2}AAAE^{L-x-1}\Omega_P] \\ &\quad - \Gamma^{0\,1}_{1\,0}\,\mathrm{Tr}[E^{x-1}BE^{-1}BEAE^{L-x-2}\Omega_P] - \Gamma^{2\,0}_{0\,2}\,\mathrm{Tr}[E^{x-2}BE^{-1}BEAE^{L-x-1}\Omega_P] \\ &\quad - \Gamma^{1\,2}_{2\,1}\,\mathrm{Tr}[E^{x-1}ABE^{-1}BE^{L-x-1}\Omega_P] + \left(3\Gamma^{0\,0}_{0\,0} - \Gamma^{0\,2}_{0\,2} - \Gamma^{2\,1}_{2\,1} - \Gamma^{1\,0}_{1\,0}\right) \\ &\quad \times \mathrm{Tr}[E^{x-1}BE^{-1}BAE^{L-x-1}\Omega_P] \end{aligned} \tag{62}$$

and for amplitudes $f(x, x + 1, x + 1)$

$$\begin{aligned}\varepsilon_3 \mathrm{Tr}[E^{x-1}ABE^{-1}BE^{L-x-1}\Omega_P] &= -\Gamma^{1\,1}_{2\,0}\,\mathrm{Tr}[E^{x-1}AAAE^{L-x-2}\Omega_P] \\ &\quad - \Gamma^{1\,0}_{0\,1}\,\mathrm{Tr}[E^{x-2}AEBE^{-1}BE^{L-x-1}\Omega_P] - \Gamma^{0\,2}_{2\,0} \\ &\quad \times \mathrm{Tr}[E^{x-1}AEBE^{-1}BE^{L-x-2}\Omega_P] - \Gamma^{2\,1}_{1\,2}\mathrm{Tr}[E^{x-1}BE^{-1}BAE^{L-x-1}\Omega_P] \\ &\quad + \left(3\Gamma^{0\,0}_{0\,0} - \Gamma^{2\,0}_{2\,0} - \Gamma^{1\,2}_{1\,2} - \Gamma^{0\,1}_{0\,1}\right)\mathrm{Tr}[E^{x-1}ABE^{-1}BE^{L-x-1}\Omega_P]. \end{aligned} \tag{63}$$

Inserting (48), (50) and (53) with $n = 3$ in (62) and (63) we obtain the new algebraic relations relating three spectral parameter matrices. Using (57) and (58) to simplify those expressions we obtain

$$\sum_{q,r,s=1}^{3} \left(D_1(k_q, k_r, k_s) B_{k_q} B_{k_r} A_{k_s} + \Gamma^{1\,1}_{2\,0}\, \mathrm{e}^{\mathrm{i}(k_r+k_s)} A_{k_q} A_{k_r} A_{k_s} - \Gamma^{1\,2}_{2\,1}\, \mathrm{e}^{\mathrm{i}(k_r+k_s)} A_{k_q} B_{k_r} B_{k_s}\right) = 0 \tag{64}$$

$$\sum_{q,r,s=1}^{3} \left(D_2(k_q, k_r, k_s) A_{k_q} B_{k_r} B_{k_s} + \Gamma^{1\,1}_{0\,2}\, \mathrm{e}^{\mathrm{i}k_s} A_{k_q} A_{k_r} A_{k_s} - \Gamma^{2\,1}_{1\,2}\, \mathrm{e}^{\mathrm{i}k_s} A_{k_q} B_{k_r} B_{k_s}\right) = 0 \tag{65}$$

where

$$\begin{aligned} D_1(k_1, k_2, k_3) &= \Gamma^{1\,0}_{0\,1} + \mathrm{e}^{\mathrm{i}(k_1+k_2+k_3)}\Gamma^{0\,2}_{2\,0} - \mathrm{e}^{\mathrm{i}k_3}\left(\Gamma^{0\,0}_{0\,0} + \Gamma^{2\,1}_{2\,1} - \Gamma^{0\,1}_{0\,1} - \Gamma^{2\,0}_{2\,0}\right) \\ D_2(k_1, k_2, k_3) &= \Gamma^{2\,0}_{0\,2} + \mathrm{e}^{\mathrm{i}(k_1+k_2+k_3)}\Gamma^{0\,1}_{1\,0} - \mathrm{e}^{\mathrm{i}(k_2+k_3)}\left(\Gamma^{0\,0}_{0\,0} + \Gamma^{1\,2}_{1\,2} - \Gamma^{1\,0}_{1\,0} - \Gamma^{0\,2}_{0\,2}\right). \end{aligned} \tag{66}$$

Differently from the $n = 2$ case the new relations (65) and (66) are in general not consistent with relations (57) and (58). This will restrict the integrability of the Hamiltonian (39) on special manifolds of the coupling constants Γ^{kl}_{mo}.

As in the $n = 2$ case, the use of (60), with $n = 3$, implies that an arbitrary amplitude of $|\Psi_{3,P}\rangle$ is proportional to $\mathrm{Tr}\left[A_{k_l} A_{k_j} A_{k_m} E^L \Omega_P\right]$. The cyclic property of the trace supplemented by the algebraic relations (43), (49) and (59) give us the relation that fixes, the up to now free, complex spectral parameters $\{k_1, k_2, k_3\}$, i.e.,

$$\mathrm{e}^{\mathrm{i}k_j L} = -\prod_{l=1}^{n} S(k_j, k_l) \qquad (j = 1, \ldots, n) \tag{67}$$

with $n = 3$ and $S(k_j, k_l)$ given by (59).

$n = 4$. The relations coming from the amplitudes with collisions with up to three particles are solved by (48) and (53), with energy and momentum given by (50). The algebraic relations obtained are generalizations of (57), (58), (64) and (65). A new relation relating the product of four matrices comes from the amplitude $f(x, x, x+1, x+1)$ where four particles are at colliding positions:

$$\begin{aligned}\varepsilon_4 \mathrm{Tr}[E^{x-1}BE^{-1}BBE^{-1}BE^{L-x-1}\Omega_P] &= -\Gamma^{1\,1}_{0\,2}\,\mathrm{Tr}[E^{x-2}AABE^{-1}BE^{L-x-1}\Omega_P] \\ &\quad - \Gamma^{1\,1}_{2\,0}\,\mathrm{Tr}[E^{x-1}BE^{-1}BAAE^{L-x-2}\Omega_P] \\ &\quad - \Gamma^{2\,0}_{0\,2}\,\mathrm{Tr}[E^{x-2}BE^{-1}BEBE^{-1}BE^{L-x-1}\Omega_P] \\ &\quad - \Gamma^{0\,2}_{2\,0}\,\mathrm{Tr}[E^{x-1}BE^{-1}BEBE^{-1}BE^{L-x-2}\Omega_P] \\ &\quad + \left(3\Gamma^{0\,0}_{0\,0} - \Gamma^{0\,2}_{0\,2} - \Gamma^{2\,0}_{2\,0} - \Gamma^{2\,2}_{2\,2}\right)\mathrm{Tr}[E^{x-1}BE^{-1}BBE^{-1}BE^{L-x-1}\Omega_P]. \end{aligned} \tag{68}$$

Inserting (48), (50) and (53) and using (57) and (58) in this last expression we obtain the algebraic relation with four spectral parameter matrices

$$\begin{aligned}\sum_{q,r,s,t=1}^{4} &\left(D_4(k_q,k_r,k_s,k_t)B_{k_q}B_{k_r}B_{k_s}B_{k_t}\right. \\ &\left. + \Gamma^{1\,1}_{2\,0}\,\mathrm{e}^{\mathrm{i}(k_r+k_s+k_t)}A_{k_q}A_{k_r}B_{k_s}B_{k_t} + \Gamma^{1\,1}_{0\,2}\,\mathrm{e}^{\mathrm{i}k_t}B_{k_q}B_{k_r}A_{k_s}A_{k_t}\right) = 0 \end{aligned} \tag{69}$$

where

$$D_4(k_1,k_2,k_3,k_4) = \Gamma^{2\,0}_{0\,2} + \Gamma^{0\,2}_{2\,0}\,\mathrm{e}^{\mathrm{i}(k_1+k_2+k_3+k_4)} - \left(\Gamma^{2\,2}_{2\,2} + \Gamma^{0\,0}_{0\,0} - \Gamma^{0\,2}_{0\,2} - \Gamma^{2\,0}_{2\,0}\right)\mathrm{e}^{\mathrm{i}(k_3+k_4)}. \tag{70}$$

This relation will impose, in addition to (64) and (65), further restrictions for the coupling constants Γ^{kl}_{mo} of the general Hamiltonian (39). The spectral parameters $(k_1, \ldots, k_4)$, as in the previous cases, are fixed by the cyclic property of the trace, and are given by (67) with $n = 4$.

General $n > 4$. All the amplitudes produce relations that are solved by the spectral parameter matrices introduced in (48) and (53). These matrices $\{A_{k_j}\}$ and $\{B_{k_j}\}$ $(j = 1, \ldots, n)$ should obey the set of algebraic relations

$$\left(A_{k_j}\right)^2 = \left(B_{k_j}\right)^2 = 0 \qquad (j = 1, \ldots, n) \tag{71}$$

$$\sum_{p^{(2)}} N(k_{p_1}, k_{p_2})A_{k_{p_1}}A_{k_{p_2}} = \sum_{p^2} C_1\left(k_{p_1}, k_{p_2}\right)B_{k_{p_1}}B_{k_{p_2}} \qquad (j \neq l = 1, \ldots, n) \tag{72}$$

$$\sum_{p^{(2)}} C_0\left(k_{p_1}, k_{p_2}\right)B_{k_{p_1}}B_{k_{p_2}} = \sum_{p_2} C_2\left(k_{p_1}, k_{p_2}\right)\mathrm{e}^{\mathrm{i}k_{p_1}}A_{k_{p_1}}A_{k_{p_2}} \qquad (j \neq l = 1, \ldots, n) \tag{73}$$

$$\begin{aligned}\sum_{p^{(3)}} &\left(D_1\left(k_{p_1}, k_{p_2}, k_{p_3}\right)B_{k_{p_1}}B_{k_{p_2}}A_{k_{p_3}} + \Gamma^{1\,1}_{2\,0}\,\mathrm{e}^{\mathrm{i}(k_{p_2}+k_{p_3})}A_{k_{p_1}}A_{k_{p_2}}A_{k_{p_3}}\right. \\ &\left. - \Gamma^{1\,2}_{2\,1}\,\mathrm{e}^{\mathrm{i}(k_{p_2}+k_{p_3})}A_{k_{p_1}}B_{k_{p_2}}B_{k_{p_3}}\right) = 0 \end{aligned} \tag{74}$$

$$\sum_{p^{(3)}} \left(D_2(k_{p_1}, k_{p_2}, k_{p_3})A_{k_{p_1}}B_{k_{p_2}}B_{k_{p_3}} + \Gamma^{1\,1}_{0\,2}\,\mathrm{e}^{\mathrm{i}k_{p_3}}A_{k_{p_1}}A_{k_{p_2}}A_{k_{p_3}} - \Gamma^{1\,2}_{2\,1}\,\mathrm{e}^{\mathrm{i}k_{p_3}}A_{k_{p_1}}B_{k_{p_2}}B_{k_{p_3}}\right) = 0 \tag{75}$$

$$\begin{aligned}\sum_{p^{(4)}} &\left(D_4\left(k_{p_1}, k_{p_2}, k_{p_3}, k_{p_4}\right)B_{k_{p_1}}B_{k_{p_2}}B_{k_{p_3}}B_{k_{p_4}} + \Gamma^{1\,1}_{2\,0}\,\mathrm{e}^{\mathrm{i}(k_{p_2}+k_{p_3}+k_{p_4})}A_{k_{p_1}}A_{k_{p_2}}B_{k_{p_3}}B_{k_{p_4}}\right. \\ &\left. + \Gamma^{1\,1}_{0\,2}\,\mathrm{e}^{\mathrm{i}k_{p_4}}B_{k_{p_1}}B_{k_{p_2}}A_{k_{p_3}}A_{k_{p_4}}\right) = 0 \end{aligned} \tag{76}$$

where in the above expressions the sums are over the permutations $p^{(m)}$ $(m = 2, 3, 4)$ $\{p_1, \ldots, p_m\}$ of m distinct integers $(j_1, \ldots, j_m)$ taken from the set $(1, 2, \ldots, n)$. Since the general Hamiltonian (39) has only nearest-neighbour interactions we have no new relations. In fact only (74)–(76) produce constraints for the integrability of (39). Exploring the fact that $C_1(k, k')$ is symmetric through the interchange $k \leftrightarrow k'$ we can use (72) to eliminate the matrices $\{B_k\}$ in (74)–(76):

$$\sum_{p^{(3)}} \left\{ D_1\left(k_{p_1}, k_{p_2}, k_{p_3}\right) N\left(k_{p_1}, k_{p_2}\right) C_1\left(k_{p_2}, k_{p_3}\right) + \mathrm{e}^{\mathrm{i}(k_{p_2}+k_{p_3})} \left[\Gamma^{1\,1}_{2\,0} C_1\left(k_{p_2}, k_{p_3}\right) - \Gamma^{1\,2}_{2\,1} N\left(k_{p_2}, k_{p_3}\right) \right] C_1\left(k_{p_1}, k_{p_2}\right) \right\} C_1\left(k_{p_1}, k_{p_3}\right) A_{k_{p_1}} A_{k_{p_2}} A_{k_{p_3}} = 0 \tag{77}$$

$$\sum_{p^{(3)}} \left\{ D_2\left(k_{p_1}, k_{p_2}, k_{p_3}\right) N\left(k_{p_2}, k_{p_3}\right) C_1\left(k_{p_1}, k_{p_2}\right) + \mathrm{e}^{\mathrm{i}k_{p_3}} \left[\Gamma^{1\,1}_{0\,2} C_1\left(k_{p_1}, k_{p_2}\right) - \Gamma^{2\,1}_{1\,2} N\left(k_{p_1}, k_{p_2}\right) \right] \right\} \times C_1\left(k_{p_2}, k_{p_2}\right) C_1\left(k_{p_1}, k_{p_3}\right) A_{k_{p_1}} A_{k_{p_2}} A_{k_{p_3}} = 0 \tag{78}$$

$$\sum_{p^{(4)}} \left\{ D_4\left(k_{p_1}, k_{p_2}, k_{p_3}, k_{p_4}\right) N\left(k_{p_1}, k_{p_2}\right) N\left(k_{p_3}, k_{p_4}\right) + \Gamma^{1\,1}_{2\,0}\, \mathrm{e}^{\mathrm{i}(k_{p_2}+k_{p_3}+k_{p_4})} C_1\left(k_{p_1}, k_{p_2}\right) N\left(k_{p_3}, k_{p_4}\right) + \Gamma^{1\,1}_{0\,2}\, \mathrm{e}^{\mathrm{i}k_{p_4}} N\left(k_{p_1}, k_{p_2}\right) C_1(k_{p_3}, k_{p_4}) \right\} C_1\left(k_{p_1}, k_{p_3}\right) C_1\left(k_{p_1}, k_{p_4}\right) \times C_1\left(k_{p_2}, k_{p_3}\right) C_1\left(k_{p_2}, k_{p_4}\right) A_{k_{p_1}} A_{k_{p_2}} A_{k_{p_3}} A_{k_{p_4}} = 0. \tag{79}$$

The above relations should be consistent with the commutation relations

$$A_{k_j} A_{k_l} = S(k_j, k_l) A_{k_l} A_{k_j} \qquad (j \neq l) \qquad \left(A_{k_j}\right)^2 = 0 \qquad (j, l = 1, 2, \ldots, n) \tag{80}$$

for any values of $k_j \in C$ $(j = 1, \ldots, 4)$. The use of (80) in (77)–(79) enables us to express the left-hand side as a polynomial on the variables $\mathrm{e}^{\mathrm{i}k_{p_j}}$ $(j = 1, \ldots, 4)$. A sufficient condition for the integrability is obtained by requiring that all coefficients of this polynomial are zero.

Although we did not consider the problem of finding all the possible solutions of (77)–(79) with (80) we verified that all the known exactly integrable spin-1 chains with a single conservation law are solutions of these equations, defining properly an associative algebra. These are the cases of the following models. The Fateev–Zamolodchikov model [30] where

$$\begin{aligned}
&\Gamma^{0\,0}_{0\,0} = \Gamma^{2\,2}_{2\,2} = 0 && \Gamma^{0\,1}_{0\,1} = \Gamma^{2\,1}_{2\,1} = \Gamma^{2\,0}_{0\,2} = \Gamma^{0\,2}_{2\,0} = -1 \\
&\Gamma^{1\,0}_{0\,1} = \Gamma^{0\,1}_{1\,0} = \Gamma^{2\,1}_{1\,2} = \Gamma^{1\,2}_{2\,1} = 1 && \Gamma^{1\,1}_{0\,2} = \Gamma^{0\,2}_{1\,1} = \Gamma^{2\,0}_{1\,1} = \Gamma^{1\,1}_{2\,0} = 2\cos(\gamma) \\
&\Gamma^{0\,2}_{0\,2} = \Gamma^{2\,0}_{2\,0} = -3 + 4\sin^2(\gamma) && \Gamma^{1\,0}_{1\,0} = \Gamma^{1\,2}_{1\,2} = -1 + 4\sin^2(\gamma) \\
&\Gamma^{1\,1}_{1\,1} = -2 + 4\sin^2(\gamma)
\end{aligned} \tag{81}$$

and γ is a free parameter; the Izergin–Korepin model [31] where

$$\begin{aligned}
&\Gamma^{1\,0}_{0\,1} = \Gamma^{0\,1}_{1\,0} = \Gamma^{1\,2}_{2\,1} = \Gamma^{2\,1}_{1\,2} = 1 && \Gamma^{0\,0}_{0\,0} = \Gamma^{0\,1}_{0\,1} = \Gamma^{1\,0}_{1\,0} = 0 \\
&\Gamma^{2\,0}_{0\,2} = \Gamma^{0\,2}_{2\,0} = \frac{\cosh(\gamma)}{\cosh(3\gamma)} && \Gamma^{2\,0}_{1\,1} = \Gamma^{1\,1}_{2\,0} = \frac{\cosh(2\gamma)}{\cosh(3\gamma)}\, \mathrm{e}^{2\gamma} \\
&\Gamma^{1\,1}_{0\,2} = \Gamma^{0\,2}_{1\,1} = -\frac{\cosh(2\gamma)}{\cosh(3\gamma)}\, \mathrm{e}^{-2\gamma} && \Gamma^{1\,1}_{1\,1} = 2\frac{\cosh(\gamma)\cosh(2\gamma)}{\cosh(3\gamma)} \\
&\Gamma^{0\,2}_{0\,2} = \Gamma^{2\,0}_{2\,0} = \Gamma^{2\,0}_{0\,2} + 2\Gamma^{1\,1}_{1\,1} \sinh^2(\gamma) && \Gamma^{1\,2}_{1\,2} = \Gamma^{0\,2}_{0\,2} + \Gamma^{2\,0}_{0\,2}\, \mathrm{e}^{-4\gamma} \\
&\Gamma^{2\,1}_{2\,1} = \Gamma^{0\,2}_{0\,2} + \Gamma^{2\,0}_{0\,2}\, \mathrm{e}^{4\gamma} && \Gamma^{2\,2}_{2\,2} = 2\left(\Gamma^{0\,2}_{0\,2} + \cosh(4\gamma)\Gamma^{2\,0}_{0\,2}\right)
\end{aligned} \tag{82}$$

and γ is also a free parameter; and the spin-1 Hamiltonian introduced in [32] where

$$\begin{aligned}
&\Gamma^{0\,0}_{0\,0} = \Gamma^{0\,1}_{0\,1} = \Gamma^{1\,0}_{1\,0} = 0 && \Gamma^{1\,0}_{0\,1} = \Gamma^{0\,1}_{1\,0} = -1 && \Gamma^{2\,0}_{0\,2} = \Gamma^{0\,2}_{2\,0} = -t \\
&\Gamma^{2\,1}_{1\,2} = \Gamma^{1\,2}_{2\,1} = -\varepsilon && \Gamma^{1\,1}_{1\,1} = \varepsilon t && \Gamma^{2\,2}_{2\,2} = -\frac{2-\varepsilon}{t} \\
&\Gamma^{2\,0}_{1\,1} = \Gamma^{1\,1}_{2\,0} = -\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\sqrt{t^2-1} && \Gamma^{1\,1}_{0\,2} = \Gamma^{0\,2}_{1\,1} = \varepsilon\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\sqrt{t^2-1} && \\
&\Gamma^{1\,2}_{1\,2} = -\frac{1}{2}\left(\frac{2-\varepsilon}{t} - \mathrm{i}\varepsilon\sqrt{3}t\right) && \Gamma^{2\,1}_{2\,1} = -\frac{1}{2}\left(\frac{2-\varepsilon}{t} + \mathrm{i}\varepsilon\sqrt{3}t\right) && \\
&\Gamma^{0\,2}_{0\,2} = \Gamma^{2\,0}_{2\,0} = -\frac{1}{2}\left(\frac{2-\varepsilon}{t} + \varepsilon t\right) &&&&
\end{aligned} \tag{83}$$

with $\varepsilon = \pm 1$ and t a free parameter. In all these cases the eigenenergies and momentum are given by (50) with $\{k_j\}$ given by (67) with the appropriate $S(k_j, k_l)$ matrix given in (59), in agreement with the Bethe ansatz solutions of these models. In order to conclude this section we mention that analogously to the generalized XXZ presented in section 2, we can also produce straightforwardly a generalization of the models (81)–(83) where the spin-1 and spin-0 particles (s^z-basis) have now an effective size $s(1, 2, \ldots)$ distinct from the size 1 of the particles with spin -1. A choice of the matrices as in (27), i.e.,

$$A^{(s)} = \sum_{j=1}^{n} A^{(s)}_{k_j} E^{2-s} \qquad B^{(s)} = \sum_{j=1}^{n} B^{(s)}_{k_j} E^{2-s} \tag{84}$$

will give the same relations derived previously except that the spectral parameter, instead of been fixed by (67) is now given by

$$\mathrm{e}^{\mathrm{i}k_j L} = \prod_{l=1, l\neq j}^{n} \left(\frac{\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}k_l}}\right)^{s-1} S(k_j, k_l) \qquad (j = 1, \ldots, n). \tag{85}$$

4. Models of spin 1 with two conservation laws

In this section, we are going to formulate our matrix product ansatz for models describing the dynamics of two types of particles on the lattice, where the total number of particles of each type is conserved separately. Integrable models in this category are the spin-1 quantum chains such as the anisotropic Sutherland model [33] or Perk–Schultz model [34] and the t–J model [35], and the stochastic Hamiltonian that emerges from the problem of asymmetric diffusion of particles hierarchically ordered [43]. Similarly as we did in section 1 in order to illustrate the versatility of our matrix product ansatz we are going to derive the extensions of the above models for the case where the two types of particles (types 1 and 2), or $s_z = 1$ and $s_z = 0$ in the spin representation of the models have arbitrary hard-core interactions. Particles of species 1 (2) will have an effective size s_1 (s_2) that excludes the presence of particles on its rightmost $s_1 - 1$ sites ($s_2 - 1$ sites), where $s_1, s_2 = 1, 2, \ldots$. The above-mentioned integrable models correspond to the particular case where $s_1 = s_2 = 1$.

Let us attach to each site $i = 1, 2, \ldots, L$ of the lattice a site variable Q_i that takes the values $Q_i = 0$ if the site is empty or the values $Q_i = 1$, $Q_i = 2$ if the site is occupied by a particle of type 1 (size s_1), type 2 (size s_2), respectively. The general Hamiltonian we consider has a $U(1)\times$ U(1) symmetry and governs the fluctuations of the configurations $\{Q_1, \ldots, Q_L\}$

on a ring of perimeter L:

$$H^{U(1)\times U(1)}_{s_1,s_2} = -\mathcal{P}\sum_{j=1}^{L}\left[\sum_{\alpha=1}^{2}\left(\Gamma^{\alpha\ 0}_{0\ \alpha}E^{0,\alpha}_{j}E^{\alpha,0}_{j+1}+\Gamma^{0\ \alpha}_{\alpha\ 0}E^{\alpha,0}_{j}E^{0,\alpha}_{j+1}\right)\right.$$
$$\left.+\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\Gamma^{\alpha\ \beta}_{\beta\ \alpha}E^{\beta,\alpha}_{j}E^{\alpha,0}_{j+s_\beta}E^{0,\beta}_{j+s_\alpha}+\sum_{\alpha=0}^{2}\sum_{\beta=0}^{2}\Gamma^{\alpha\ \beta}_{\alpha\ \beta}E^{\alpha,\alpha}_{j}E^{\beta,\beta}_{j+s_\alpha}\right]\mathcal{P}+L\Gamma^{0\ 0}_{0\ 0} \tag{86}$$

where $E^{\alpha,\beta}$ $(\alpha,\beta = 0,1,2)$ are the usual 3×3 Weyl matrices with i,j elements $\left(E^{\alpha,\beta}\right)_{i,j} = \delta_{\alpha,i}\delta_{\beta,j}$ and $\Gamma^{\gamma\ \delta}_{\alpha\ \beta}$ are the coupling constants. The projector $\mathcal{P}$ in (86) projects out from the space of configurations those where the particles occupy forbidden positions due to their sizes. The last sum in (86) accounts for the 'static' interactions while the first and second sums are the 'kinetic' terms representing the motion and interchange of particles, respectively. The $U(1)\times U(1)$ symmetry supplemented by the periodic boundary condition of (86) implies that the total number of particles $n_1, n_2 = 0,1,2,\ldots$ on classes 1 and 2 as well as the momentum $P = \frac{2\pi l}{L}$ $(l = 0,1,\ldots,L-1)$ are good quantum numbers.

We want to formulate a matrix product ansatz for the eigenvectors $|\Psi_{n_1,n_2,P}\rangle$ belonging to the eigensector labelled by (n_1, n_2, P) and satisfying the eigenvalue equation

$$H^{U(1)\times U(1)}_{s_1,s_2}\left|\Psi_{n_1,n_2,P}\right\rangle = \varepsilon_{n_1,n_2}\left|\Psi_{n_1,n_2,P}\right\rangle. \tag{87}$$

These eigenvectors are given by

$$\left|\Psi_{n_1,n_2,P}\right\rangle = \sum_{\{Q\}}\sum_{\{x\}} f(x_1,Q_1;\ldots;x_n,Q_n)|x_1,Q_1;\ldots;x_n,Q_n\rangle \tag{88}$$

where the kets $|x_1,Q_1;\ldots;x_n,Q_n\rangle$ denote the configurations with particles of type Q_i $(Q_i = 1,2)$ located at the positions x_i $(x_i = 1,\ldots,L)$. The total number of particles is $n = n_1 + n_2$. The summation $\{Q\} = \{Q_1,\ldots,Q_n\}$ extends over all the sequences of n numbers $\{1,2\}$ in which n_1 terms have value 1 and n_2 terms the value 2, while the summation $\{x\} = \{x_1,\ldots,x_n\}$ extends, for each sequence $\{Q\}$, into the set of the nondecreasing integers satisfying

$$x_{i+1} \geqslant x_i + s_{Q_i} = 1,\ldots,n-1 \qquad s_{Q_1} \leqslant x_n - x_1 \leqslant L - s_{Q_n}. \tag{89}$$

The matrix product ansatz we propose asserts that the amplitudes of an arbitrary eigenfunction (88) are given in terms of traces of the matrix product

$$f(x_1,Q_1;\ldots;x_n,Q_n) = \mathrm{Tr}[E^{x_1-1}Y^{(Q_1)}E^{x_2-x_1-1}Y^{(Q_2)}\cdots E^{x_n-x_{n-1}-1}Y^{(Q_n)}E^{L-x_n}\Omega_P]. \tag{90}$$

The matrices $Y^{(Q)}$ are associated with the particles of type Q $(Q = 1,2)$. As in the previous sections the matrix E is associated with the vacant sites and the matrix Ω_P satisfying

$$E\Omega_P = \mathrm{e}^{-\mathrm{i}P}\Omega_P E \qquad Y^{(Q)}\Omega_P = \mathrm{e}^{-\mathrm{i}P}\Omega_P Y^{(Q)} \qquad (Q = 1,2) \tag{91}$$

ensures (see (7)) the momentum $P = \frac{2\pi}{L}l$ $(l = 0,1,\ldots,L-1)$ of the eigenvector. Let us consider initially the cases of a small number of particles. Since for $n = 1$ we have a similar situation as in the previous sections let us consider the case where $n = 2$.

$n = 2$. For two particles of types Q_1 and Q_2 $(Q_1, Q_2 = 1,2)$ on the lattice we have two types of relations. The eigenvalue equation applied to the components where the particles of class

Q_1 and Q_2 are at positions (x_1, x_2) with $x_2 > x_1 + s_{Q_1}$ gives us

$$\begin{aligned}\varepsilon^{(Q_1,Q_2)}\,&\mathrm{Tr}[E^{x_1-1}Y^{(Q_1)}E^{x_2-x_1-1}Y^{(Q_2)}E^{L-x_2}\Omega_P]\\ &= -\Gamma^{Q_1\,0}_{0\,Q_1}\mathrm{Tr}[E^{x_1-2}Y^{(Q_1)}E^{x_2-x_1}Y^{(Q_2)}E^{L-x_2}\Omega_P]\\ &\quad -\Gamma^{0\,Q_1}_{Q_1\,0}\mathrm{Tr}[E^{x_1}Y^{(Q_1)}E^{x_2-x_1-2}Y^{(Q_2)}E^{L-x_2}\Omega_P]\\ &\quad -\Gamma^{Q_2\,0}_{0\,Q_2}\mathrm{Tr}[E^{x_1-1}Y^{(Q_1)}E^{x_2-x_1-2}Y^{(Q_2)}E^{L-x_2+1}\Omega_P]\\ &\quad -\Gamma^{0\,Q_2}_{Q_2\,0}\mathrm{Tr}[E^{x_1-1}Y^{(Q_1)}E^{x_2-x_1}Y^{(Q_2)}E^{L-x_2-1}\Omega_P] + \big(4\Gamma^{0\,0}_{0\,0} - \Gamma^{0\,Q_1}_{0\,Q_1}\\ &\quad -\Gamma^{Q_1\,0}_{Q_1\,0} - \Gamma^{0\,Q_2}_{0\,Q_2} - \Gamma^{Q_2\,0}_{Q_2\,0}\big)\mathrm{Tr}[E^{x_1-1}Y^{(Q_1)}E^{x_2-x_1-1}Y^{(Q_2)}E^{L-x_2}\Omega_P] \end{aligned}\tag{92}$$

while the components where the particles are at the colliding positions $(x_1 = x, x_2 = x_1 + s_{Q_1})$ give us

$$\begin{aligned}\varepsilon^{(Q_1,Q_2)}\,&\mathrm{Tr}[E^{x-1}Y^{(Q_1)}E^{s_{Q_1}-1}Y^{(Q_2)}E^{L-x-s_{Q_1}}\Omega_P]\\ &= -\Gamma^{Q_1\,0}_{0\,Q_1}\mathrm{Tr}[E^{x-2}Y^{(Q_1)}E^{s_{Q_1}}Y^{(Q_2)}E^{L-x-s_{Q_1}}\Omega_P]\\ &\quad -\Gamma^{0\,Q_2}_{Q_2\,0}\mathrm{Tr}[E^{x-1}Y^{(Q_1)}E^{s_{Q_1}}Y^{(Q_2)}E^{L-x-s_{Q_1}-1}\Omega_P]\\ &\quad -\Gamma^{Q_2\,Q_1}_{Q_1\,Q_2}\mathrm{Tr}[E^{x-1}Y^{(Q_2)}E^{s_{Q_2}-1}Y^{(Q_1)}E^{L-x-s_{Q_2}}\Omega_P]\\ &\quad +\big(3\Gamma^{0\,0}_{0\,0} - \Gamma^{0\,Q_1}_{0\,Q_1} - \Gamma^{Q_2\,0}_{Q_2\,0} - \Gamma^{Q_1\,Q_2}_{Q_1\,Q_2}\big)\\ &\quad \times \mathrm{Tr}[E^{x-1}Y^{(Q_1)}E^{s_{Q_1}-1}Y^{(Q_2)}E^{L-x-s_{Q_1}}\Omega_P]. \end{aligned}\tag{93}$$

Let us consider initially the case where the particles are of the same type. In this case, we have exactly the same situation as in the XXZ chain considered in section 1, and a solution of (92) and (93) is obtained by identifying $Y^{(Q)}$ as composed by two spectral parameter dependent matrices $Y^{(Q)}_{k_1}$ and $Y^{(Q)}_{k_2}$, i.e.,

$$Y^{(Q)} = \sum_{j=1}^{n} Y^{(Q)}_{k_j}E^{2-s_Q} \qquad \text{with} \quad EY^{(Q)}_{k_j} = \mathrm{e}^{\mathrm{i}k_j}Y^{(Q)}_{k_j}E \qquad \big(Y^{(Q)}_{k_j}\big)^2 = 0 \qquad (Q = 1, 2)\tag{94}$$

with $n = 2$. These last relations, when inserted in (92), give us the energy and momentum in terms of the spectral parameters k_j $(j = 1, 2)$

$$\varepsilon^{(Q,Q)} = \sum_{j=1}^{n}\varepsilon^{(Q)}(k_j) \qquad P = \sum_{j=1}^{n}k_j\tag{95}$$

where $n = 2$ and $\varepsilon^{(Q)}(k)$ is given by

$$\varepsilon^{(Q)}(k) = -\big(\Gamma^{Q\,0}_{0\,Q}\,\mathrm{e}^{-\mathrm{i}k} + \Gamma^{0\,Q}_{Q\,0}\,\mathrm{e}^{\mathrm{i}k} - 2\Gamma^{0\,0}_{0\,0} + \Gamma^{0\,Q}_{0\,Q} + \Gamma^{Q\,0}_{Q\,0}\big) \qquad (Q = 1, 2).\tag{96}$$

Using (94) and (95) in (93) we obtain the relations

$$Y^{(Q)}_{k_j}Y^{(Q)}_{k_l} = S^{Q\,Q}_{Q\,Q}(k_j, k_l)Y^{(Q)}_{k_l}Y^{(Q)}_{k_j} \qquad (j \neq l) \qquad \big(Y^{(Q)}_{k_j}\big)^2 = 0 \qquad (1 \leqslant j, l \leqslant n)\tag{97}$$

where

$$S^{Q\,Q}_{Q\,Q}(k_j, k_l) = -\frac{\Gamma^{Q\,0}_{0\,Q} + \Gamma^{0\,Q}_{Q\,0}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} - \big(\Gamma^{0\,0}_{0\,0} + \Gamma^{Q\,Q}_{Q\,Q} - \Gamma^{Q\,0}_{Q\,0} - \Gamma^{0\,Q}_{0\,Q}\big)\,\mathrm{e}^{\mathrm{i}k_j}}{\Gamma^{Q\,0}_{0\,Q} + \Gamma^{0\,Q}_{Q\,0}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} - \big(\Gamma^{0\,0}_{0\,0} + \Gamma^{Q\,Q}_{Q\,Q} - \Gamma^{Q\,0}_{Q\,0} - \Gamma^{0\,Q}_{0\,Q}\big)\,\mathrm{e}^{\mathrm{i}k_l}} \qquad (Q = 1, 2).\tag{98}$$

Let us consider now the case where the particles are of distinct species. The distinguibility of the particles allows two types of solutions of (92) and (93). We may try a standard solution as in (94) (case a) where each of the matrices $Y^{(Q)}$ $(Q = 1, 2)$ is composed of two spectral

parameter matrices, with the same values of the spectral parameters k_1, k_2 (see (94)) or alternatively (case b) we may consider a special solution where each $Y^{(Q)}$ is composed of a single spectral parameter matrix, with a distinct spectral parameter, i.e.,

$$Y^{(1)} = Y^{(1)}_{k_1} E^{2-s_1} \qquad Y^{(2)} = Y^{(2)}_{k_2} E^{2-s_2} \qquad \text{with} \quad E Y^{(j)}_{k_j} = \mathrm{e}^{\mathrm{i}k_j} Y^{(j)}_{k_j} E \quad (j = 1, 2). \tag{99}$$

In case b, (92) gives us the energy and momentum as in (95) while (93) gives us two independent equations:

$$\begin{aligned}\big[-\big(\Gamma^{Q_2\,0}_{0\,Q_2} + \Gamma^{0\,Q_1}_{Q_1\,0}\,\mathrm{e}^{\mathrm{i}(k_1+k_2)}\big) + \big(\Gamma^{0\,0}_{0\,0} + \Gamma^{Q_1\,Q_2}_{Q_1\,Q_2} - \Gamma^{Q_1\,0}_{Q_1\,0} - \Gamma^{0\,Q_2}_{0\,Q_2}\big)\mathrm{e}^{\mathrm{i}k_{Q_2}}\big] Y^{(Q_1)}_{k_{Q_1}} Y^{(Q_2)}_{k_{Q_2}} \\ + \Gamma^{Q_2\,Q_1}_{Q_1\,Q_2}\,\mathrm{e}^{\mathrm{i}k_{Q_2}} Y^{(Q_2)}_{k_{Q_2}} Y^{(Q_1)}_{k_{Q_1}} = 0 \qquad (Q_1 \neq Q_2 = 1, 2).\end{aligned} \tag{100}$$

Since at this level we want to keep k_1 and k_2 as free complex parameters (100) implies special choices of the coupling constants Γ^{mo}_{kl} of (86), for example[3]

$$\Gamma^{2\,0}_{0\,2} = \Gamma^{0\,1}_{1\,0} = \Gamma^{2\,1}_{1\,2} = 0 \qquad \Gamma^{0\,0}_{0\,0} + \Gamma^{1\,2}_{1\,2} = \Gamma^{1\,0}_{1\,0} + \Gamma^{0\,2}_{0\,2} \tag{101}$$

that gives

$$Y^{(1)}_{k_1} Y^{(2)}_{k_2} = S^{1\,2}_{1\,2}(k_1, k_2) Y^{(2)}_{k_2} Y^{(1)}_{k_1} \qquad Y^{(2)}_{k_2} Y^{(1)}_{k_1} = S^{2\,1}_{2\,1}(k_2, k_1) Y^{(1)}_{k_1} Y^{(2)}_{k_2} \tag{102}$$

where

$$S^{1\,2}_{1\,2}(k_1, k_2) = \frac{1}{S^{2\,1}_{2\,1}(k_2, k_1)} = \frac{\Gamma^{1\,0}_{0\,1} + \Gamma^{0\,2}_{2\,0}\,\mathrm{e}^{\mathrm{i}(k_1+k_2)} - \big(\Gamma^{0\,0}_{0\,0} + \Gamma^{2\,1}_{2\,1} - \Gamma^{2\,0}_{2\,0} - \Gamma^{0\,1}_{0\,1}\big)\,\mathrm{e}^{\mathrm{i}k_1}}{\Gamma^{1\,2}_{2\,1}\,\mathrm{e}^{\mathrm{i}k_1}}. \tag{103}$$

Let us consider case a. A solution of (92) and (93) where $Y^{(Q)}$ $(Q = 1, 2)$ is given in terms of two spectral parameter matrices $\big\{Y^{(Q)}_{k_j}\big\}$ as in (94) is possible only if these matrices satisfy

$$Y^{(Q_1)}_{k_j} Y^{(Q_2)}_{k_j} = 0 \qquad j = 1, 2 \quad (Q_1, Q_2 = 1, 2) \tag{104}$$

and the Hamiltonian (86) has its coupling constants restricted to

$$\Gamma^{1\,0}_{0\,1} = \Gamma^{2\,0}_{0\,2} \qquad \Gamma^{0\,1}_{1\,0} = \Gamma^{0\,2}_{2\,0}. \tag{105}$$

In this case the energy and momentum are given by

$$\varepsilon^{(Q_1,Q_2)} = \varepsilon^{(Q_1)}(k_1) + \varepsilon^{(Q_2)}(k_2) \qquad P = k_1 + k_2 \tag{106}$$

respectively. Inserting (94) and (106) into (93) we obtain algebraic relations that can be written in a matrix form

$$\sum_{l,m=1}^{2} \begin{bmatrix} \mathcal{D}_{l,m} + v_{Q_2,Q_1}\,\mathrm{e}^{\mathrm{i}k_m} & \Gamma^{Q_2\,Q_1}_{Q_1\,Q_2}\,\mathrm{e}^{\mathrm{i}k_m} \\ \Gamma^{Q_1\,Q_2}_{Q_2\,Q_1}\,\mathrm{e}^{\mathrm{i}k_m} & \mathcal{D}_{l,m} + v_{Q_1,Q_2}\,\mathrm{e}^{\mathrm{i}k_m} \end{bmatrix} \begin{bmatrix} Y^{(Q_1)}_{k_l} Y^{(Q_2)}_{k_m} \\ Y^{(Q_2)}_{k_l} Y^{(Q_1)}_{k_m} \end{bmatrix} = 0. \tag{107}$$

where

$$\begin{aligned}\mathcal{D}_{l,m} &= -\big(\Gamma^{Q_1\,0}_{0\,Q_1} + \Gamma^{0\,Q_1}_{Q_1\,0}\,\mathrm{e}^{\mathrm{i}(k_l+k_m)}\big) \\ v_{Q_2,Q_1} &= \Gamma^{0\,0}_{0\,0} + \Gamma^{Q_1\,Q_2}_{Q_1\,Q_2} - \Gamma^{Q_1\,0}_{Q_1\,0} - \Gamma^{0\,Q_2}_{0\,Q_2} \qquad (Q_1, Q_2 = 1, 2).\end{aligned} \tag{108}$$

This last relation can be rearranged straightforwardly (see, e.g., [43]) giving us $(Q_1 \neq Q_2)$

$$Y^{(Q_1)}_{k_l} Y^{(Q_2)}_{k_m} = S^{Q_1\,Q_2}_{Q_1\,Q_2}(k_l, k_m) Y^{(Q_2)}_{k_m} Y^{(Q_1)}_{k_l} + S^{Q_1\,Q_2}_{Q_2\,Q_1}(k_l, k_m) Y^{(Q_1)}_{k_m} Y^{(Q_2)}_{k_l} \tag{109}$$

[3] Another equivalent choice is obtained by the interchange $1 \leftrightarrow 2$ in (101)–(103).

where

$$S^{Q_1\ Q_2}_{Q_2\ Q_1}(k_l,k_m) = -\left\{1 - \frac{\mathrm{e}^{\mathrm{i}k_l} - \mathrm{e}^{k_m}}{\Delta}\left[\left(\mathcal{D}_{l,m} + v_{Q_1,Q_2}\,\mathrm{e}^{\mathrm{i}k_m}\right)v_{Q_2,Q_1} - \Gamma^{Q_2\ Q_1}_{Q_1\ Q_2}\Gamma^{Q_1\ Q_2}_{Q_2\ Q_1}\,\mathrm{e}^{\mathrm{i}k_m}\right]\right\}$$
$$S^{Q_1\ Q_2}_{Q_1\ Q_2}(k_l,k_m) = -\left\{1 - \frac{\mathrm{e}^{\mathrm{i}k_l} - \mathrm{e}^{k_m}}{\Delta}\left[\left(\mathcal{D}_{l,m} + v_{Q_1,Q_2}\,\mathrm{e}^{\mathrm{i}k_m}\right)\Gamma^{Q_2\ Q_1}_{Q_1\ Q_2} - \Gamma^{Q_2\ Q_1}_{Q_1\ Q_2}v_{Q_1,Q_2}\,\mathrm{e}^{\mathrm{i}k_m}\right]\right\} \tag{110}$$

and

$$\Delta = \left(\mathcal{D}_{l,m} + v_{Q_2,Q_1}\,\mathrm{e}^{\mathrm{i}k_m}\right)\left(\mathcal{D}_{l,m} + v_{Q_1,Q_2}\,\mathrm{e}^{\mathrm{i}k_m}\right) - \Gamma^{Q_2\ Q_1}_{Q_1\ Q_2}\Gamma^{Q_1\ Q_2}_{Q_2\ Q_1}\,\mathrm{e}^{\mathrm{i}2k_m}. \tag{111}$$

In all cases, for $Q_1 = Q_2$ (see (97) and (98)) or $Q_1 \neq Q_2$ in case b (see (101)–(103)) or case a (see (109)–(111)) the cyclic property of the traces in our matrix product ansatz (90) will fix the spectral parameters. Instead of producing these equations for $n = 2$ let us consider the case of general values of n.

General n. We now consider the case of arbitrary numbers n_1, n_2 of particles of types 1 and 2 ($n = n_1 + n_2$). The eigenvalue equation (87) when applied to the components of the eigenfunction $\left|\Psi_{n_1,n_2,P}\right\rangle$ where all the particles are not at colliding positions, gives us a generalization of (92) that can be solved in two distinct ways as in the case $n = 2$. In case b, where the coupling constants satisfy (101), we identify the matrices $Y^{(Q)}$ ($Q = 1, 2$) as composed by the spectral dependent matrices

$$Y^{(1)} = \sum_{j=1}^{n_1} Y^{(1)}_{k_j} E^{2-s_1} \qquad Y^{(2)} = \sum_{j=n_1+1}^{n} Y^{(2)}_{k_j} E^{2-s_2} \tag{112}$$

where $Y^{(Q)}_{k_j}$ satisfy

$$\begin{aligned} &EY^{(Q)}_{k_j} = \mathrm{e}^{\mathrm{i}k_j}Y^{(Q)}_{k_j}E \qquad \left(Y^{(Q)}_{k_j}\right)^2 = 0 \\ &n_{Q-1} < j \leqslant n_Q + (Q-1)n_{Q-1} \qquad (Q = 1, 2) \end{aligned} \tag{113}$$

with $n_0 = 0$. On the other hand in case a, where the coupling constants satisfy (105), the matrices $Y^{(Q)}$ are given by

$$Y^{(Q)} = \sum_{j=1}^{n} Y^{(Q)}_{k_j} E^{2-s_Q} \qquad \text{with} \qquad EY^{(Q)}_{k_j} = \mathrm{e}^{\mathrm{i}k_j}Y^{(Q)}_{k_j}E \qquad 1 \leqslant j \leqslant n. \tag{114}$$

In cases a and b the energy and momentum, in terms of the spectral parameters $\{k_j\}$, are given by

$$\varepsilon_{n_1,n_2} = \sum_{j=1}^{n_1} \varepsilon^{(1)}(k_j) + \sum_{j=n_1+1}^{n} \varepsilon^{(2)}(k_j) \qquad P = \sum_{j=1}^{n} k_j. \tag{115}$$

The amplitudes of $\left|\Psi_{n_1,n_2,P}\right\rangle$ where a pair of particles of types Q_1 and Q_2 are located at the closest positions x_i and $x_{i+1} = x_i + s_{Q_1}$ will give us the following algebraic relations. In case b we obtain

$$\begin{aligned} &Y^{(Q_1)}_{k_j}Y^{(Q_2)}_{k_l} = S^{Q_1\ Q_2}_{Q_1\ Q_2}(k_j,k_l)Y^{(Q_2)}_{k_l}Y^{(Q_1)}_{k_j} \qquad (Q_1, Q_2 = 1, 2) \quad (k_j \neq k_l) \\ &n_{Q_1-1} < j \leqslant n_{Q_1} + (Q_1-1)n_{Q_1-1} \qquad n_{Q_2-1} < l \leqslant n_{Q_2} + (Q_2-1)n_{Q_2-1} \qquad n_0 = 0 \end{aligned} \tag{116}$$

where the algebraic structure constants are the diagonal S matrix with non-zero elements given by

$$S^{Q\,Q}_{Q\,Q}(k_j,k_l) = -\frac{\Gamma^{Q\,0}_{0\,Q} + \Gamma^{0,Q}_{Q,0}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} - \left(\Gamma^{0\,0}_{0\,0} + \Gamma^{Q\,Q}_{Q\,Q} - \Gamma^{Q\,0}_{Q\,0} - \Gamma^{0\,Q}_{0\,Q}\right)\mathrm{e}^{\mathrm{i}k_j}}{\Gamma^{Q\,0}_{0\,Q} + \Gamma^{0,Q}_{Q,0}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} - \left(\Gamma^{0\,0}_{0\,0} + \Gamma^{Q\,Q}_{Q\,Q} - \Gamma^{Q\,0}_{Q\,0} - \Gamma^{0\,Q}_{0\,Q}\right)\mathrm{e}^{\mathrm{i}k_l}}$$
$$S^{1\,2}_{1\,2}(k_j,k_l) = \frac{1}{S^{2\,1}_{2\,1}(k_l,k_j)} = \frac{\Gamma^{1\,0}_{0\,1} + \Gamma^{0\,2}_{2\,0}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} - \left(\Gamma^{0\,0}_{0\,0} + \Gamma^{2\,1}_{2\,1} - \Gamma^{2\,0}_{2\,0} - \Gamma^{0\,1}_{0\,1}\right)\mathrm{e}^{\mathrm{i}k_j}}{\Gamma^{1\,2}_{2\,1}\,\mathrm{e}^{\mathrm{i}k_j}}. \tag{117}$$

In case a, where the coupling constants satisfy (105) we obtain

$$Y^{(Q_1)}_{k_l} Y^{(Q_2)}_{k_m} = \sum_{Q_1'=1}^{2}\sum_{Q_2'=1}^{2} S^{Q_1\,Q_2}_{Q_2'\,Q_1'}(k_l,k_m) Y^{(Q_1')}_{k_m} Y^{(Q_2')}_{k_l} \qquad (k_l \neq k_m) \quad 1 \leqslant l,m \leqslant n \tag{118}$$

where the non-vanishing values of the non-diagonal S matrix are the six values given in (110) and (98) with the choice (101).

For arbitrary amplitudes we have in our matrix product ansatz (90) a product of n matrices $\left\{Y^{(Q)}_{k_j}\right\}$. Our ansatz will be valid only if the relations (116) in case b or (118) in case a provide a unique relation among these products, otherwise $\left|\Psi_{n_1,n_2,P}\right\rangle$ is not properly defined. This means, for example, that the products $\cdots Y^{(\alpha)}_{k_1} Y^{(\beta)}_{k_2} Y^{(\gamma)}_{k_3} \cdots$ and $\cdots Y^{(\gamma)}_{k_3} Y^{(\beta)}_{k_2} Y^{(\alpha)}_{k_1} \cdots$ should be uniquely related. Since we can relate then either by performing the commutations in the order $\alpha\beta\gamma \to \beta\alpha\gamma \to \beta\gamma\alpha \to \gamma\beta\alpha$ or $\alpha\beta\gamma \to \alpha\gamma\beta \to \gamma\alpha\beta \to \gamma\beta\alpha$ the structure constants $S^{\alpha\,\alpha'}_{\gamma\,\gamma'}$ of the algebraic relations (116) and (118) should satisfy

$$\sum_{\gamma,\gamma',\gamma''=1}^{2} S^{\alpha\,\alpha'}_{\gamma\,\gamma'}(k_1,k_2) S^{\gamma\,\alpha''}_{\beta\,\gamma''}(k_1,k_3) S^{\gamma'\,\gamma''}_{\beta'\,\beta''}(k_2,k_3)$$
$$= \sum_{\gamma,\gamma',\gamma''=1}^{2} S^{\alpha'\,\alpha''}_{\gamma'\,\gamma''}(k_2,k_3) S^{\alpha\,\gamma''}_{\gamma\,\beta''}(k_1,k_3) S^{\gamma\,\gamma'}_{\beta\,\beta'}(k_1,k_2) \tag{119}$$

for $\alpha,\alpha',\alpha'',\beta,\beta',\beta'' = 1,2$. This last constraint is just the Yang–Baxter relation [2, 44] of the S matrix defined in (116) and (118). Actually the condition (119) is enough to ensure that any matrix product of spectral matrices $\left\{Y^{(Q)}_{k_j}\right\}$ is uniquely related and it implies the associativity of the algebra of the operators $\left\{Y^{(Q)}_{k_j}\right\}$.

In case b, the S-matrix given by (117) is diagonal and the Yang–Baxter relation (119) is satisfied trivially, so that the only restriction in the coupling constants of the Hamiltonian (86) is given by (101). In case a the S matrix given in (98) and (110) is non-diagonal and the Yang–Baxter relation (119) produces strong constraints in the allowed couplings of the Hamiltonian (86).

We did not consider in this paper the problem of finding all the possible coupling constants $\Gamma^{\gamma\,\delta}_{\alpha\,\beta}$ in (86) that renders the algebra (118) associative, or equivalently, to find all the solutions of the Yang–Baxter relations (119). It is important to stress that our matrix product ansatz (90) produce the same relations (118) independently of the hard-core sizes s_1 and s_2 of the particles of types 1 and 2, respectively. Solutions of (119) were presented in the literature previously. The solution where

$$\Gamma^{\alpha\,\beta}_{\beta\,\alpha} = 1 \qquad (\alpha \neq \beta) \qquad \Gamma^{\alpha\,\alpha}_{\alpha\,\alpha} = \epsilon_\alpha \cosh(\gamma)$$
$$\Gamma^{\alpha\,\beta}_{\alpha\,\beta} = \mathrm{sign}(\alpha-\beta)\sinh(\gamma) \qquad (\alpha \neq \beta) \tag{120}$$

where γ is a free parameter and $\epsilon_1,\epsilon_2,\epsilon_3 = \pm 1$, corresponds to the anisotropic Perk–Schultz model [34]. The isotropic model obtained by setting $\gamma = 0$ and $\epsilon_1,\epsilon_2,\epsilon_3 = \pm 1$ is the $SU(3)$

Sutherland model [33]. The solution (120) with $\epsilon_1 = -\epsilon_2 = \epsilon_3 = 1$ gives us the anisotropic supersymmetric t–J model[4]. A solution of (119) is also known [47] for the case where

$$\Gamma^{\alpha\,\beta}_{\beta\,\alpha} = -\Gamma^{\alpha\,\beta}_{\alpha\,\beta} = q^{\mathrm{sign}(\alpha-\beta)} \quad (\alpha \neq \beta) \qquad \Gamma^{\alpha\,\alpha}_{\alpha\,\alpha} = 0 \quad (\alpha, \beta = 0, 1, 2) \tag{121}$$

with q real and $q \geqslant 1$. The Hamiltonian (86) with the couplings (121) describes the time fluctuations in the asymmetric diffusion problem of two species of particles hierarchically ordered (asymmetry parameter q). A generalization of this problem, treated through the matrix product ansatz, is presented elsewhere [45].

In order to complete our solutions through the matrix product ansatz (90) we should fix the spectral parameters $(k_1, \ldots, k_n)$. Using the algebraic relations (113) or (114) an arbitrary amplitude is proportional to

$$\mathrm{Tr}\big[Y^{(1)}_{k_1} \cdots Y^{(1)}_{k_{n_1}} Y^{(2)}_{k_{n_1+1}} \cdots Y^{(2)}_{k_n} E^{L-n_1(s_1-1)-n_2(s_2-1)} \Omega_P\big]. \tag{122}$$

The cyclic property of the trace fixes the spectral parameters.

Let us consider case b, where the coupling constants satisfy (101). The commutation relations (91), (113) and (116) give us

$$\begin{aligned} \mathrm{e}^{\mathrm{i}k_j L} &= \left[\prod_{l=1}^{n_1} S^{1\,1}_{1\,1}(k_j, k_l) \left(\frac{\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}k_l}}\right)^{s_1-1}\right] \prod_{l=n_1+1}^{n} S^{1\,2}_{1\,2}(k_j, k_l) \left(\frac{\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}k_l}}\right)^{s_2-1} \qquad (1 \leqslant j \leqslant n_1) \\ \mathrm{e}^{\mathrm{i}k_j L} &= \left[\prod_{l=n_1+1}^{n} S^{2\,2}_{2\,2}(k_j, k_l) \left(\frac{\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}k_l}}\right)^{s_2-1}\right] \prod_{l=1}^{n_1} S^{2\,1}_{2\,1}(k_j, k_l) \left(\frac{\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}k_l}}\right)^{s_1-1} \qquad (n_1 < j \leqslant n) \end{aligned} \tag{123}$$

where the S matrix is given by (117). The energy and momentum are obtained by inserting the solutions of (123) into (115). The Hamiltonian (86) that is exactly integrable in case b (see (101)) is given by

$$\begin{aligned} H^{(b)} = -\sum_{j=1}^{L} \mathcal{P} \big\{ & \Gamma^{1\,0}_{0\,1} E^{0\,1}_j E^{1\,0}_{j+1} + \Gamma^{0\,2}_{2\,0} E^{2\,0}_j E^{0\,2}_{j+1} + \Gamma^{1\,2}_{2\,1} E^{2\,1}_j E^{1\,0}_{j+s_2} E^{0\,2}_{j+s_1} + \Gamma^{1\,1}_{1\,1} E^{1\,1}_j E^{1\,1}_{j+s_1} \\ & + \Gamma^{2\,2}_{2\,2} E^{2\,2}_j E^{2\,2}_{j+s_2} + \Gamma^{0\,1}_{0\,1} E^{0\,0}_j E^{1\,1}_{j+1} + \Gamma^{1\,0}_{1\,0} E^{1\,1}_j E^{0\,0}_{j+s_1} + \Gamma^{0\,2}_{0\,2} E^{0\,0}_j E^{2\,2}_{j+1} \\ & + \Gamma^{2\,0}_{2\,0} E^{2\,2}_j E^{0\,0}_{j+s_2} + \Gamma^{2\,1}_{2\,1} E^{2\,2}_j E^{1\,1}_{j+s_2} + \left(\Gamma^{1\,0}_{0\,1} + \Gamma^{0\,2}_{2\,0}\right) E^{1\,1}_j E^{2\,2}_{j+s_1} \big\} \mathcal{P} \end{aligned} \tag{124}$$

where we chose $\Gamma^{0\,0}_{0\,0} = 0$ but still we have ten free parameters! The particular choice $s_1 = s_2 = 1$, $\Gamma^{1\,2}_{2\,1} = -\Gamma^{1\,2}_{1\,2} = \Gamma^{1\,0}_{0\,1} + \Gamma^{0\,2}_{2\,0}$, $\Gamma^{1\,0}_{0\,1} = -\Gamma^{1\,0}_{1\,0}$, $\Gamma^{0\,2}_{2\,0} = -\Gamma^{0\,2}_{0\,2}$, $\Gamma^{1\,1}_{1\,1} = \Gamma^{2\,2}_{2\,2} = \Gamma^{0\,1}_{0\,1} = \Gamma^{2\,0}_{2\,0} = 0$, gives the Hamiltonian related to the stochastic problem of fully asymmetric diffusion of two kinds of particles, whose exact integrability was obtained in [26] through the dynamical matrix product ansatz.

Let us return to the general case. The cyclic property of the trace in (122) and the commutation relations (91), (114) and (118) give us

$$\begin{aligned} &\mathrm{Tr}\big[Y^{(Q_1)}_{k_1} \cdots Y^{(Q_n)}_{k_n} E^{L-n_1(s_1-1)-n_2(s_2-1)} \Omega_P\big] \\ &\quad = \mathrm{e}^{\mathrm{i}k_j[L-n_1(s_1-1)-n_2(s_2-1)]} \sum_{Q'_1,\ldots,Q'_n=1}^{2} \langle Q_1, \ldots, Q_n | \mathcal{T} | Q'_1, \ldots, Q'_n \rangle \\ &\qquad \times \mathrm{Tr}\big[Y^{(Q'_1)}_{k_1} \cdots Y^{(Q'_n)}_{k_n} E^{L-n_1(s_1-1)-n_2(s_2-1)} \Omega_P\big] \end{aligned} \tag{125}$$

where we have used the identity (see (98) and (110))

$$\sum_{Q''_j, Q''_{j+1}} S^{Q_j\;Q''_{j+1}}_{Q'_j\;Q''_j}(k_j, k_j) = -1 \tag{126}$$

[4] A Jordan–Wigner fermionization of the Hamiltonian (86) with $s_1 = s_2 = 1$ and coupling constants (120) give us the anisotropic supersymmetric t–J model [35].

and

$$\langle Q_1,\ldots,Q_n|\mathcal{T}|Q'_1,\ldots,Q'_n\rangle = \sum_{Q''_1,\ldots,Q''_n}\left\{S^{Q_1\ Q''_2}_{Q'_1\ Q''_1}(k_1,k_j)\cdots S^{Q_j\ Q''_{j+1}}_{Q'_j\ Q''_j}(k_j,k_j)\cdots S^{Q_n\ Q''_1}_{Q'_n\ Q''_n}(k_n,k_j)\,\mathrm{e}^{\mathrm{i}P\left(s''_1-1\right)}\right\} \quad (127)$$

is the transfer matrix of a $2^n\times 2^n$-dimensional transfer matrix of an inhomogeneous 6-vertex model (inhomogeneities $\{k_l\}$) with Boltzmann weights given by (98) and (110). The 6-vertex model is defined on a cylinder of transversal perimeter n with a seam along its axis producing the twisted boundary conditions

$$S^{Q_nQ''_{n+1}}_{Q'_nQ''_n}(k_n,k) = S^{Q_nQ''_1}_{Q'_nQ''_n}(k_n,k)\phi(s_{Q''_1}) \quad (128)$$

where

$$\phi(s) = \mathrm{e}^{\mathrm{i}P(s-1)} \quad (129)$$

and, as before, P is the momentum of the eigenstate (88). The relation (125) gives us the constraints for the spectral parameters

$$\exp(-\mathrm{i}k_j(L+n-n_1s_1-n_2s_2)) = \Lambda(k_j,\{k_l\}) \qquad (j=1,\ldots,n) \quad (130)$$

where $\Lambda(k_j,\{k_l\})$ are the eigenvalues of the transfer matrix (127). The condition (130) leads to the problem of evaluation of the eigenvalues of the inhomogeneous transfer matrix (127). This can be done through the algebraic Bethe ansatz [46] or the coordinate Bethe ansatz (see [43] and [47] for example), and we obtain

$$\Lambda(k_j,\{k_l\}) = \phi(s_2)\prod_{l=1}^{n}S^{2\,2}_{2\,2}(k_l,k_j)\prod_{l=1}^{n_1}\frac{S^{2\,2}_{2\,2}\left(k_j,k_l^{(1)}\right)}{S^{2\,1}_{2\,1}\left(k_j,k_l^{(1)}\right)} \quad (131)$$

where the unknown parameters $k_l^{(1)}$ $(l=1,\ldots,n_1)$ are fixed by the n_1 coupled equations

$$\frac{\phi(s_1)}{\phi(s_2)}\prod_{l=1}^{n}\frac{S^{2\,1}_{2\,1}\left(k_l,k_j^{(1)}\right)}{S^{2\,2}_{2\,2}\left(k_l,k_j^{(1)}\right)} = \prod_{l=1}^{n_1}\frac{S^{2\,2}_{2\,2}\left(k_j^{(1)},k_l^{(1)}\right)}{S^{1\,1}_{1\,1}\left(k_l^{(1)},k_j^{(1)}\right)}\frac{S^{2\,1}_{2\,1}\left(k_l^{(1)},k_j^{(1)}\right)}{S^{2\,1}_{2\,1}\left(k_j^{(1)},k_l^{(1)}\right)} \quad (132)$$

for $j=1,\ldots,n_1$. Equations (130)–(132) fix the spectral parameters $\{k_j;\ j=1,\ldots,n\}$ of the matrices $Y^{(Q)}_{k_j}$ introduced in our matrix product ansatz.

5. Models of spin-$\frac{3}{2}$ with two conservation laws

As in the previous section we now formulate our matrix product ansatz for $U(1)\times U(1)$ symmetric models describing the dynamics of two types of particles on the lattice, where the total number of particles of each type is conserved separately. However now, distinct from the last section, particles of distinct types may occupy the same site on the lattice (double occupancy). Models in this category are the spin-$\frac{3}{2}$ anisotropic Perk–Schultz model [34], the Essler–Korepin–Schoutens model [36], the Hubbard model [37], the Hamiltonian derived in [48] from the R matrix introduced in [49], and the two-parameter integrable model introduced in [38].

In order to define the Hamiltonians for these models on a lattice with L sites let us attach to each lattice site a variable, Q_i $(i=1,\ldots,L)$, that takes the values $Q_i=0$ if the site is empty, $Q_i=1,2$ if the site is occupied by a particle of type 1 or 2, respectively, and $Q_i=3$ if the site has a double occupancy of particles of types 1 and 2. In the fermionic version of the models the particles of types 1 and 2 are the electrons with distinct spin polarizations. The

most general model with nearest-neighbour interactions and periodic boundary condition is given by

$$H^{U(1)\times U(1)} = -\sum_{j=1}^{L}\left(\sum_{\alpha\neq\beta=0}^{3}\sum_{\gamma\neq\nu=0}^{3}\Gamma_{\gamma\ \nu}^{\alpha\ \beta}E_j^{\gamma\ \alpha}E_{j+1}^{\nu\ \beta}+\sum_{\alpha=0}^{3}\sum_{\beta=1}^{3}\Gamma_{\alpha\ \beta}^{\alpha\ \beta}E_j^{\alpha\ \alpha}E_{j+1}^{\beta\ \beta}\right) \tag{133}$$

where the coupling constants satisfy $\Gamma_{\gamma\ \nu}^{\alpha\ \beta}=0$ if $\alpha+\beta\neq\gamma+\nu$, $\Gamma_{\beta\ \gamma}^{\alpha\ \alpha}=\Gamma_{\alpha\ \alpha}^{\beta\ \gamma}=0$ unless $\alpha=\beta=\gamma$, and $E^{\alpha\ \beta}$ ($\alpha,\beta=0,1,2,3$) are the 4×4 Weyl matrices with i,j elements $(E^{\alpha\ \beta})_{i,j}=\delta_{\alpha,i}\delta_{\beta,j}$. Without losing any generality we also chose hereafter $\Gamma_{0\ 0}^{0\ 0}=0$. The first and second summations in (133) account for the kinetic and static interactions. The $U(1)\times U(1)$ symmetry and the translation symmetry, due to the periodic boundary condition of (133), imply that the total number of particles n_1, n_2 $(0,1,\ldots)$ of particles of types 1 and 2 and the momentum $P=\frac{2\pi l}{L}$ ($l=0,1,\ldots,L-1$) are conserved separately.

We formulate a matrix product ansatz for the eigenvectors $\left|\Psi_{n_1,n_2,P}\right\rangle$ of the eigenvalue equation

$$H^{U(1)\times U(1)}\left|\Psi_{n_1,n_2,P}\right\rangle=\varepsilon_{n_1,n_2}\left|\Psi_{n_1,n_2,P}\right\rangle \tag{134}$$

in the eigensector (n_1,n_2,P). An arbitrary eigenfunction is given by

$$\left|\Psi_{n_1,n_2,P}\right\rangle=\sum_{\{Q\}}\sum_{\{x\}}f(x_1,Q_1;\ldots;x_n,Q_n)|x_1,Q_1;\ldots;x_n,Q_n\rangle \tag{135}$$

where $|x_1,Q_1;\ldots;x_n,Q_n\rangle$ are the configurations whose particles of type $Q_i=1,2$ are located at $x_i=1,\ldots,L$ and $n=n_1+n_2$. The summation $\{Q\}=\{Q_1,\ldots,Q_n\}$ extends over all permutations of n numbers $\{1,2\}$ in which n_1 terms have the value 1 and n_2 terms the value 2. The summation $\{x\}=\{x_1,\ldots,x_n\}$ extends, for each permutation $\{Q\}$, into the set of non-decreasing integers where $(x_{i+1}-x_i)\geqslant\delta_{Q_{i+1},Q_i}$, allowing double occupancy of sites by distinct type of particles.

In order to formulate a matrix product ansatz we associate with the sites occupied by $Q_i=0$, 1 and 2 the matrices E, $Y^{(1)}$ and $Y^{(2)}$, respectively. In analogy to the results of section 3 we associate with the double occupied sites ($Q_i=3$) the matrix product $Y^{(3)}=B^{(1)}E^{-1}B^{(2)}$. Our ansatz asserts that the amplitudes corresponding to the configurations with no double occupied sites are given by the traces

$$f(x_1,Q_1;\ldots;x_n,Q_n)=\mathrm{Tr}[E^{x_1-1}Y^{(Q_1)}E^{x_2-x_1-1}Y^{(Q_2)}\cdots E^{x_n-x_{n-1}-1}Y^{(Q_n)}E^{L-x_n}\Omega_P] \tag{136}$$

while the amplitudes related to configurations with double occupancy at $x_{i+1}=x_i$ are given by the traces

$$\begin{aligned}f(x_1,Q_1;\ldots;x_i,1;x_i,2;\ldots;x_n,Q_n)&\\ =\mathrm{Tr}\left[E^{x_1-1}Y^{(Q_1)}\cdots E^{x_i-x_{i-1}-1}B^{(1)}E^{-1}B^{(2)}E^{x_{i+1}-x_i-1}\cdots E^{L-x_n}\Omega_P\right].&\end{aligned} \tag{137}$$

The general cases are given by generalizations of (136) and (137).

Similarly as in the previous sections the matrix Ω_P satisfying

$$\begin{aligned}&E\Omega_P=\mathrm{e}^{-\mathrm{i}P}\Omega_P E \qquad && Y^{(Q)}\Omega_P=\mathrm{e}^{-\mathrm{i}P}\Omega_P Y^{(Q)}\\ &B^{(Q)}\Omega_P=\mathrm{e}^{-\mathrm{i}P}\Omega_P B^{(Q)} && (Q=1,2)\end{aligned} \tag{138}$$

ensures (see (7)) the momentum P of the eigenvector $\left|\Psi_{n_1,n_2,P}\right\rangle$.

As in the previous sections in order to satisfy the eigenvalue equation (134) we identify $Y^{(Q)}$ and $B^{(Q)}$ $(Q = 1, 2)$ as composed of n spectral parameter dependent matrices[5]

$$Y^{(Q)} = \sum_{j=1}^{n} Y_{k_j}^{(Q)} E \qquad B^{(Q)} = \sum_{j=1}^{n} B_{k_j}^{(Q)} E (Q = 1, 2) \tag{139}$$

satisfying the commutation relations

$$\begin{aligned} &Y_{k_j}^{(Q)} E = \mathrm{e}^{\mathrm{i}k_j} E Y_{k_j}^{(Q)} \qquad B_{k_j}^{(Q)} E = \mathrm{e}^{\mathrm{i}k_j} E B_{k_j}^{(Q)} \\ &\left[Y_{k_j}^{(Q)}, \Omega_P\right] = \left[B_{k_j}^{(Q)}, \Omega_P\right] = Y_{k_j}^{(Q)} Y_{k_j}^{(Q')} = B_{k_j}^{(Q)} B_{k_j}^{(Q')} = 0 \qquad (Q, Q' = 1, 2) \end{aligned} \tag{140}$$

where k_j $(j = 1, \ldots, n)$ are unknown complex spectral parameters.

The eigenvalue equation (134) when applied to the components of $\left|\Psi_{n_1,n_2,P}\right\rangle$ with no particles at colliding positions $(x_{i+1} > x_i + 1, i = 1, \ldots, n)$ implies the same constraint (105) and gives us a generalization of (115), namely

$$\begin{aligned} &\varepsilon_{n_1,n_2} = \sum_{j=1}^{n_1} \varepsilon^{(1)}(k_j) + \sum_{j=n_1+1}^{n} \varepsilon^{(2)}(k_j) \qquad P = \sum_{j=1}^{n} k_j \\ &\varepsilon^{Q}(k) = -\left(\Gamma_{0\,Q}^{Q\,0}\,\mathrm{e}^{-\mathrm{i}k} + \Gamma_{Q\,0}^{0\,Q}\,\mathrm{e}^{\mathrm{i}k} + \Gamma_{Q\,0}^{0\,Q} + \Gamma_{Q\,0}^{Q\,0}\right) \qquad Q = 1, 2. \end{aligned} \tag{141}$$

Let us now consider the amplitudes of $|\Psi_{n_1,n_2,P}\rangle$ where a pair of particles $(i, i+1)$ of types Q_i and Q_{i+1} are located at the colliding positions $x_{i+1} = x_i$ or $x_{i+1} = x_i + 1$. The cases where $Q_i = Q_{i+1} = Q$ and $Q_i \neq Q_{i+1}$ give us distinct relations. If $Q_i = Q_{i+1}$ we have a similar situation as that considered in the previous section and we obtain $S_{Q\,Q}^{Q\,Q}(k_j, k_l)$ as in (98) with $\Gamma_{0\,0}^{0\,0} = 0$. If $Q_i \neq Q_{i+1}$ we have distinct relations merging from the configurations where $x_i = x_{i+1}$ and $x_{i+1} = x_i + 1$. Inserting our ansatz (139) and using (141) and (140) these equations are given by

$$\sum_{j,l=1\,j\neq l}^{n} C_0(k_j, k_l) B_{k_j}^{(1)} B_{k_l}^{(2)} = -\sum_{j,l=1\,j\neq l}^{n} \mathrm{e}^{\mathrm{i}k_l}\left[C_1'(k_j, k_l) Y_{k_j}^{(1)} Y_{k_l}^{(2)} + C_2'(k_j, k_l) Y_{k_j}^{(2)} Y_{k_l}^{(1)}\right] \tag{142}$$

$$\begin{aligned} \sum_{j,l=1\,j\neq l}^{n} C_{Q_1}(k_j, k_l) B_{k_j}^{(1)} B_{k_l}^{(2)} = &\sum_{j,l=1\,j\neq l}^{n} \left\{\left[D(k_j, k_l) - \left(\Gamma_{Q_1\,0}^{Q_1\,0} + \Gamma_{0\,Q_2}^{0\,Q_2} - \Gamma_{Q_1\,Q_2}^{Q_1\,Q_2}\right)\mathrm{e}^{\mathrm{i}k_l}\right] Y_{k_j}^{(Q_1)} Y_{k_l}^{(Q_2)} \right. \\ &\left. + \Gamma_{Q_1\,Q_2}^{Q_2\,Q_1}\,\mathrm{e}^{\mathrm{i}k_l} Y_{k_j}^{(Q_2)} Y_{k_l}^{(Q_1)}\right\} \qquad Q_1 \neq Q_2 \quad (Q_1 = 1, 2) \end{aligned} \tag{143}$$

where C_0, C_1', $C_1^{x\prime}$, C_1, C_2' and D are the symmetric functions

$$\begin{aligned} C_0(k_j, k_l) &= -\Gamma_{0\,1}^{1\,0}(\mathrm{e}^{\mathrm{i}k_j x} + \mathrm{e}^{\mathrm{i}k_l}) - \Gamma_{1\,0}^{0\,1}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)}(\mathrm{e}^{\mathrm{i}k_j} + \mathrm{e}^{\mathrm{i}k_l}) \\ &\quad - \left(\Gamma_{1\,0}^{1\,0} + \Gamma_{0\,1}^{0\,1} + \Gamma_{2\,0}^{2\,0} + \Gamma_{0\,2}^{0\,2} - \Gamma_{0\,3}^{0\,3} - \Gamma_{3\,0}^{3\,0}\right)\mathrm{e}^{\mathrm{i}(k_j+k_l)} + \Gamma_{0\,3}^{3\,0} + \Gamma_{3\,0}^{0\,3}\,\mathrm{e}^{\mathrm{i}2(k_j+k_l)} \\ C_Q(k_j, k_l) &= -\Gamma_{Q\,Q'}^{3\,0} - \Gamma_{Q\,Q'}^{0\,3}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} \\ C_Q'(k_j, k_l) &= \Gamma_{0\,3}^{Q\,Q'} + \Gamma_{3\,0}^{Q\,Q'}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} \\ D(k_j, k_l) &= -\Gamma_{0\,1}^{1\,0} - \Gamma_{1\,0}^{0\,1}\,\mathrm{e}^{\mathrm{i}(k_j+k_l)} \qquad (Q \neq Q' = 1, 2). \end{aligned} \tag{144}$$

[5] Similar to the solutions (101) in section 4 there exist special solutions, that for brevity we do not consider here, where the matrices $Y^{(1)}$, $B^{(1)}$ and $Y^{(2)}$, $B^{(2)}$ are composed of the distinct sets of spectral parameters $\{k_1, \ldots, k_{n_1}\}$ and $\{k_{n_1}, \ldots, k_{n_1+n_2}\}$, respectively.

Multiplying the two relations in (143) by $C_0(k_j, k_l)$ and using (142) we obtain the algebraic relations

$$Y_{k_l}^{(Q_i)} Y_{k_m}^{(Q_{i+1})} = \sum_{Q=1}^{2} \sum_{Q'=1}^{2} S_{Q'\ Q}^{Q_i\ Q_{i+1}}(k_l, k_m) Y_{k_m}^{(Q)} Y_{k_l}^{(Q')} \tag{145}$$

where the non-zero values of S are $S_{Q\ Q}^{Q\ Q}$ given by (98) and

$$S_{Q'\ Q}^{Q\ Q'}(k_j, k_l) = -\frac{\alpha_{l,j}^{(Q)} \alpha_{j,l}^{(Q')} - \beta_{j,l}^{(Q)} \beta_{l,j}^{(Q')}}{\alpha_{j,l}^{(Q)} \alpha_{j,l}^{(Q')} - \beta_{j,l}^{(Q)} \beta_{j,l}^{(Q')}} \qquad S_{Q\ Q'}^{Q\ Q'}(k_j, k_l) = -\frac{\beta_{l,j}^{(Q)} \alpha_{j,l}^{(Q')} - \beta_{j,l}^{(Q)} \alpha_{l,j}^{(Q')}}{\alpha_{j,l}^{(Q)} \alpha_{j,l}^{(Q')} - \beta_{j,l}^{(Q)} \beta_{j,l}^{(Q')}} \tag{146}$$

with

$$\begin{aligned} \alpha_{j,l}^{(Q)} &= -C_Q(k_j, k_l) C'_Q(k_j, k_l)\, \mathrm{e}^{\mathrm{i}k_l} - C_0(k_j, k_l)\big[D(k_j, k_l) - \big(\Gamma_{Q\ 0}^{Q\ 0} + \Gamma_{0\ Q'}^{0\ Q'} - \Gamma_{Q\ Q'}^{Q\ Q'}\big)\mathrm{e}^{\mathrm{i}k_l}\big] \\ \beta_{j,l}^{(Q)} &= -\big[C_Q(k_j, k_l) C'_{Q'}(k_j, k_l) - C_0(k_j, k_l) \Gamma_{Q\ Q'}^{Q'\ Q}\big]\, \mathrm{e}^{\mathrm{i}k_l} \qquad (Q \neq Q' = 1, 2). \end{aligned} \tag{147}$$

Multiplying (143) with $Q_1 = 1$ by C'_1 and (142) by $\Gamma_{2\ 1}^{1\ 2}$ and subtracting the obtained expressions we obtain a relation that expresses the spectral parameter matrices $B_k^{(Q)}$ in terms of $Y_k^{(Q)}$, i.e.,

$$B_{k_j}^{(1)} B_{k_l}^{(2)} = \frac{D_3(k_j, k_l)}{\rho(k_j, k_l)} Y_{k_j}^{(1)} Y_{k_l}^{(2)} \qquad (j \neq l) \tag{148}$$

where

$$\begin{aligned} D_3(k_j, k_l) &= D(k_j, k_l) - \big(\Gamma_{1\ 0}^{1\ 0} - \Gamma_{1\ 2}^{1\ 2}\big) C'_2(k_j, k_l) - C'_1(k_j, k_l) \Gamma_{1\ 2}^{2\ 1}\, \mathrm{e}^{\mathrm{i}k_l} \\ \rho(k_j, k_l) &= -\beta_{j,l}\, \mathrm{e}^{-\mathrm{i}k_l} = C_1(k_j, k_l) C'_2(k_j, k_l) - C_0(k_j, k_l) \Gamma_{1\ 2}^{2\ 1}. \end{aligned} \tag{149}$$

Since the matrices $B_{k_j}^{(Q)}$ only appear in the pairs $B_{k_j}^{(1)} B_{k_j}^{(2)}$ in our matrix product ansatz (136) and (137) the relation (148) enables us to express the amplitudes only in terms of the spectral parameter matrices $Y_{k_j}^{(Q)}$ $(Q = 1, 2)$. Moreover our ansatz will be valid only if products such as $\cdots Y_{k_1}^{(Q_1)} Y_{k_2}^{(Q_2)} Y_{k_3}^{(Q_3)} \cdots$ and $\cdots Y_{k_3}^{(Q_3)} Y_{k_2}^{(Q_2)} Y_{k_1}^{(Q_1)} \cdots$ are uniquely related. Similarly as in the previous section this requirement implies that the structure contants $S_{Q_1 Q_2}^{Q'_1 Q'_2}$ satisfy the relation (119) that coincides with the Yang–Baxter relation [29, 2] of the S matrix defined in (98) and (146). These relations will impose severe constraints among the coupling constants $\Gamma_{\alpha\ \beta}^{\gamma\ \delta}$ of our general Hamiltonian (133). However differently from the last section the Yang–Baxter relations (119) are not enough to ensure that the eigenfunctions $\big|\Psi_{n_1,n_2,P}\big\rangle$ are given by our matrix product ansatz, since new relations among the spectral parameter matrices $\big\{Y_{k_j}^{(Q)}\big\}$ occur when we have on the lattice three or four particles at matching conditions. These new relations occur because within the range of the nearest-neighbour interactions of the Hamiltonian (133) we may have up to four particles. The eigenvalue equation (134) applied to the amplitudes where we have a particle Q at x and a pair at $x + 1$, and to the amplitude with a particle Q at $x + 1$ and a pair at x gives us, after the use of (139), (143) and (141), the following algebraic relations:

$$\begin{aligned} \sum_{j,l,m=1}^{n} \big\{ & D^{(3,Q)}(k_j, k_l, k_m) Y_{k_j}^{(Q)} B_{k_l}^{(1)} B_{k_m}^{(2)} - \mathrm{e}^{\mathrm{i}k_m} \big(\Gamma_{Q\ 3}^{3\ Q} B_{k_j}^{(1)} B_{k_l}^{(2)} Y_{k_m}^{(Q)} \\ & - \Gamma_{0\ 3}^{Q'\ Q} Y_{k_j}^{(Q)} Y_{k_l}^{(Q')} Y_{k_m}^{(Q)} - \Gamma_{0\ 3}^{Q\ Q'} Y_{k_j}^{(Q)} Y_{k_l}^{(Q)} Y_{k_m}^{(Q')}\big)\big\} = 0 \end{aligned}$$

$$\sum_{j,l,m=1}^{n} \Big\{ D^{(Q,3)}(k_j,k_l,k_m) B^{(1)}_{k_j} B^{(2)}_{k_l} Y^{(Q)}_{k_m} - \exp(\mathrm{i}(k_l+k_m)) \big(\Gamma^{Q\ 3}_{3\ Q} Y^{(Q)}_{k_j} B^{(1)}_{k_l} B^{(2)}_{k_m} - \Gamma^{Q'\ Q}_{3\ 0} Y^{(Q')}_{k_j} Y^{(Q)}_{k_l} Y^{(Q)}_{k_m} - \Gamma^{Q\ Q'}_{3\ 0} Y^{(Q)}_{k_j} Y^{(Q')}_{k_l} Y^{(Q)}_{k_m} \big) \Big\} = 0 \tag{150}$$

where $Q, Q' = 1, 2$ or $Q, Q' = 2, 1$, and

$$\begin{aligned} D^{(3,Q)}(k_j,k_l,k_m) &= \Gamma^{0\ Q}_{Q\ 0} \mathrm{e}^{\mathrm{i}(k_j+k_l+k_m)} + \left(\Gamma^{Q\ 0}_{Q\ 0} + \Gamma^{0\ 3}_{0\ 3} - \Gamma^{Q\ 3}_{Q\ 3} \right) \mathrm{e}^{\mathrm{i}(k_l+k_m)} + \Gamma^{3\ 0}_{0\ 3} \\ D^{(Q,3)}(k_j,k_l,k_m) &= \Gamma^{0\ 3}_{3\ 0} \mathrm{e}^{\mathrm{i}(k_j+k_l+k_m)} + \left(\Gamma^{0\ Q}_{0\ Q} + \Gamma^{3\ 0}_{3\ 0} - \Gamma^{3\ Q}_{3\ Q} \right) \mathrm{e}^{\mathrm{i}k_m} + \Gamma^{Q\ 0}_{0\ Q}. \end{aligned} \tag{151}$$

In the case where we have two pairs of particles at sites x and $x+1$ the eigenvalue equation (134), when applied to these amplitudes, after the use of (139), (143) and (141), gives us the relations

$$\sum_{j,l,m,o=1}^{n} \Big\{ D^{(3,3)}(k_j,k_l,k_m,k_o) B^{(1)}_{k_j} B^{(2)}_{k_l} B^{(1)}_{k_m} B^{(2)}_{k_o} + \exp(\mathrm{i}(k_l+k_m+k_o)) \big(\Gamma^{2\ 1}_{3\ 0} Y^{(2)}_{k_j} Y^{(1)}_{k_l} + \Gamma^{1\ 2}_{3\ 0} Y^{(1)}_{k_j} Y^{(2)}_{k_l} \big) B^{(1)}_{k_m} B^{(2)}_{k_o} + \mathrm{e}^{\mathrm{i}k_o} B^{(1)}_{k_j} B^{(2)}_{k_l} \big(\Gamma^{2\ 1}_{0\ 3} Y^{(2)}_{k_m} Y^{(1)}_{k_o} + \Gamma^{1\ 2}_{0\ 3} Y^{(1)}_{k_m} Y^{(2)}_{k_0} \big) \Big\} = 0 \tag{152}$$

where

$$D^{(3,3)}(k_j,k_l,k_m,k_o) = \Gamma^{3\ 0}_{0\ 3} + \Gamma^{0\ 3}_{3\ 0} \mathrm{e}^{\mathrm{i}(k_j+k_l+k_m+k_o)} + \left(\Gamma^{3\ 0}_{3\ 0} + \Gamma^{0\ 3}_{0\ 3} - \Gamma^{3\ 3}_{3\ 3} \right) \mathrm{e}^{\mathrm{i}(k_m+k_o)}. \tag{153}$$

Since any pair $B^{(1)}_{k_j} B^{(2)}_{k_l}$ $(k_j \neq k_l)$ is expressed in terms of the pair $Y^{(1)}_{k_j} Y^{(2)}_{k_l}$, through (148), we rewrite the expressions (150) and (152) only in terms of the matrices $\{Y^{(Q)}_k\}$. Multiplying (150) by the symmetric function $\rho(k_j,k_l)\rho(k_l,k_m)\rho(k_m,k_j)$ we obtain

$$\begin{aligned} &\sum_{j,l,m=1}^{n} \Big\{ D^{(3,Q)}(k_j,k_l,k_m) D_3(k_l,k_m) \rho(k_j,k_l) Y^{(Q)}_{k_j} Y^{(1)}_{k_l} Y^{(2)}_{k_m} \\ &\quad - \mathrm{e}^{\mathrm{i}k_m} \Gamma^{3\ Q}_{Q\ 3} D_3(k_j,k_l) \rho(k_l,k_m) Y^{(1)}_{k_j} Y^{(2)}_{k_l} Y^{(Q)}_{k_m} \\ &\quad + \big(\Gamma^{Q'\ Q}_{0\ 3} Y^{(Q)}_{k_j} Y^{(Q')}_{k_l} Y^{(Q)}_{k_m} + \Gamma^{Q\ Q'}_{0\ 3} Y^{(Q)}_{k_j} Y^{(Q)}_{k_l} Y^{(Q')}_{k_m} \big) \\ &\quad \times \mathrm{e}^{\mathrm{i}k_m} \rho(k_j,k_l)\rho(k_l,k_m) \Big\} \rho(k_m,k_j) = 0 \\ &\sum_{j,l,m=1}^{n} \Big\{ D^{(Q,3)}(k_j,k_l,k_m) D_3(k_j,k_l) \rho(k_l,k_m) Y^{(1)}_{k_j} Y^{(2)}_{k_l} Y^{(Q)}_{k_m} \\ &\quad - \exp(\mathrm{i}(k_l+k_m)) \Gamma^{Q\ 3}_{3\ Q} D_3(k_l,k_m) \rho(k_j,k_l) Y^{(Q)}_{k_j} Y^{(1)}_{k_l} Y^{(2)}_{k_m} \\ &\quad + \big(\Gamma^{Q'\ Q}_{3\ 0} Y^{(Q')}_{k_j} Y^{(Q)}_{k_l} Y^{(Q)}_{k_m} + \Gamma^{Q\ Q'}_{3\ 0} Y^{(Q)}_{k_j} Y^{(Q')}_{k_l} Y^{(Q)}_{k_m} \big) \\ &\quad \times \exp(\mathrm{i}(k_l+k_m)) \rho(k_j,k_l)\rho(k_l,k_m) \Big\} \rho(k_m,k_j) = 0 \end{aligned} \tag{154}$$

where $(Q, Q') = (1, 2)$ or $(2, 1)$. Multiplying (152) by the symmetric combination $\rho(k_j,k_l)\rho(k_l,k_o)\rho(k_o,k_m)\rho(k_m,k_j)\rho(k_l,k_m)\rho(k_j,k_o)$ we obtain

$$\sum_{j,l,m,o=1}^{4} \Big\{ \Big[\big(D^{(3,3)}(k_j,k_l,k_m,k_o) D_3(k_j,k_l) + \Gamma^{1\,2}_{3\,0}\, \mathrm{e}^{\mathrm{i}(k_l+k_m+k_o)} \rho(k_j,k_l) \big) D_3(k_m,k_o) + \Gamma^{1\,2}_{0\,3}\, \mathrm{e}^{\mathrm{i}k_o} D_3(k_j,k_l)\rho(k_m,k_o) \Big] Y^{(1)}_{k_j} Y^{(2)}_{k_l} Y^{(1)}_{k_m} Y^{(2)}_{k_o} + \Gamma^{2\,1}_{3\,0}\, \mathrm{e}^{\mathrm{i}(k_l+k_m+k_o)} D_3(k_m,k_o)\rho(k_j,k_l) Y^{(2)}_{k_j} Y^{(1)}_{k_l} Y^{(1)}_{k_m} Y^{(2)}_{k_o} + \Gamma^{2\,1}_{0\,3}\, \mathrm{e}^{\mathrm{i}k_o} D_3(k_j,k_l)\rho(k_m,k_o) Y^{(1)}_{k_j} Y^{(2)}_{k_l} Y^{(2)}_{k_m} Y^{(1)}_{k_o} \Big\} \times \rho(k_l,k_o)\rho(k_m,k_j)\rho(k_l,k_m)\rho(k_j,k_o) = 0. \tag{155}$$

In order to have a matrix product ansatz for the Hamiltonian (133) the relations (154) and (155) should be consistent with the two words commutation relations (145), for any values of $k_j \in C$ $(j = 1, \ldots, n)$. The successive use of (145) in (154) and (155) allows us to rewrite the left-hand sides of these equations as a polynomial on the variables $\mathrm{e}^{\mathrm{i}kj}$ $(j = 1, \ldots, n)$. Since we do not want, on this level, to fix the spectral parameters $\{k_j\}$, we should impose that all the coefficients of these polynomials are zero. This will give further constraints on the coupling constant $\Gamma^{\gamma\ \delta}_{\alpha\ \beta}$ besides (105) and those imposed by the Yang–Baxter relations (119).

No additional constraint will occur. Although we did not consider here the problem of searching all the possible solutions of the general Hamiltonian satisfying (119) and (154) and (155), we verified that those equations are satisfied by several known exact integrable chains. The solution where

$$\begin{aligned} &\Gamma^{\alpha\ \beta}_{\beta\ \alpha} = 1 \qquad \Gamma^{\alpha\ \beta}_{\alpha\ \beta} = \mathrm{sign}(\alpha-\beta)\sinh(\gamma) - \epsilon_0 \cosh(\gamma) \qquad (\alpha \neq \beta) \\ &\Gamma^{\alpha\ \alpha}_{\alpha\ \alpha} = (\epsilon_\alpha - \epsilon_0)\cosh(\gamma) \qquad (\alpha, \beta = 0, 1, 2, 3) \end{aligned} \tag{156}$$

where $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$ and γ a free complex parameter gives us the anisotropic spin-$\frac{3}{2}$ Perk–Schultz model. The fermionic version, obtained by a Jordan–Wigner transformation, of the Hamiltonian (133) with the choices (156) and with $\epsilon_0 = -\epsilon_1 = -\epsilon_2 = \epsilon_3 = 1$ is the anisotropic version of the Essler–Korepin–Schoutens model.

The solution where

$$\Gamma^{\alpha\ \beta}_{\gamma\ \nu} = \delta_{\alpha+\beta,\gamma+\nu} \quad (\alpha \neq \beta, \gamma \neq \nu) \qquad \Gamma^{0\,3}_{3\,0} = \Gamma^{3\,0}_{0\,3} = 0 \qquad \Gamma^{\alpha 3}_{\alpha 3} = \Gamma^{3\alpha}_{3\alpha} = -\frac{U}{2}(1+\delta_{\alpha,3}) \tag{157}$$

with U a free parameter, gives us after a Jordan–Wigner transformation, the standard Hubbard model with Coulomb on site interaction U [37]. The solutions where the non-zero coupling constants are

$$\begin{aligned} &\Gamma^{\alpha\ 0}_{0\ \alpha} = \Gamma^{0\ \alpha}_{\alpha\ 0} = 1 \quad (\alpha \neq 0) \qquad \Gamma^{3\,1}_{1\,3} = \Gamma^{1\,3}_{3\,1} = \Gamma^{3\,2}_{2\,3} = \Gamma^{2\,3}_{3\,2} = \epsilon \qquad \Gamma^{3\,1}_{3\,1} = \Gamma^{3\,0}_{3\,0} + \Gamma^{1\,2}_{1\,2} \\ &\Gamma^{3\,0}_{1\,2} = \Gamma^{1\,2}_{3\,0} = \epsilon\Gamma^{2\,1}_{0\,3} = \epsilon\Gamma^{0\,3}_{2\,1} = \Gamma^{1\,2}_{0\,3} = \Gamma^{0\,3}_{1\,2} = \epsilon\Gamma^{3\,0}_{2\,1} = \epsilon\Gamma^{2\,1}_{3\,0} = \sin(\theta) \\ &\Gamma^{2\,1}_{1\,2} = \Gamma^{1\,2}_{2\,1} = -\epsilon\Gamma^{0\,2}_{2\,0} = -\epsilon\Gamma^{2\,0}_{0\,2} = -\frac{\epsilon}{2}\Gamma^{3\,0}_{3\,0} = \Gamma^{2\,1}_{2\,1}\,\mathrm{e}^{2\eta} = \Gamma^{1\,2}_{1\,2}\,\mathrm{e}^{-2\eta} = \cos(\theta) \\ &\Gamma^{3\,2}_{3\,2} = \Gamma^{3\,0}_{3\,0} + \Gamma^{2\,1}_{2\,1} \qquad \Gamma^{1\,3}_{1\,3} = \Gamma^{1\,2}_{1\,2} \qquad \Gamma^{2\,3}_{2\,3} = \Gamma^{2\,1}_{2\,1} \qquad \Gamma^{3\,3}_{3\,3} = \Gamma^{3\,0}_{3\,0} + \Gamma^{2\,1}_{2\,1} + \Gamma^{1\,2}_{1\,2} \end{aligned} \tag{158}$$

and

$$\begin{aligned} &\Gamma^{\alpha\ 0}_{0\ \alpha} = \Gamma^{0\ \alpha}_{\alpha\ 0} = 1 \quad (\alpha \neq 0) \qquad \Gamma^{3\,1}_{1\,3} = \Gamma^{1\,3}_{3\,1} = \Gamma^{3\,2}_{2\,3} = \Gamma^{2\,3}_{3\,2} = \epsilon \\ &\Gamma^{3\,0}_{1\,2} = \Gamma^{1\,2}_{3\,0} = \epsilon\Gamma^{2\,1}_{0\,3} = \epsilon\Gamma^{0\,3}_{2\,1} = \epsilon\,\mathrm{e}^{2\eta}\Gamma^{1\,2}_{0\,3} = \epsilon\,\mathrm{e}^{2\eta}\Gamma^{0\,3}_{1\,2} = \mathrm{e}^{-2\eta}\Gamma^{3\,0}_{2\,1} = \mathrm{e}^{-2\eta}\Gamma^{2\,1}_{3\,0} = \sin(\theta) \\ &\Gamma^{2\,1}_{1\,2} = \Gamma^{1\,2}_{2\,1} = -\epsilon\Gamma^{0\,2}_{2\,0} = -\epsilon\Gamma^{2\,0}_{0\,2} = \Gamma^{2\,1}_{2\,1}\mathrm{e}^{2\eta} = \Gamma^{1\,2}_{1\,2}\mathrm{e}^{-2\eta} = \cos(\theta) \\ &\Gamma^{3\,0}_{3\,0} = 2\Gamma^{2\,0}_{0\,2} + \sin^2(\theta)\frac{(\mathrm{e}^{\eta} - \epsilon\,\mathrm{e}^{-\eta})^2}{\cos(\theta)} \qquad \Gamma^{3\,1}_{3\,1} = \Gamma^{3\,2}_{3\,2} = \Gamma^{3\,0}_{3\,0} + \Gamma^{1\,2}_{1\,2} \\ &\Gamma^{1\,3}_{1\,3} = \Gamma^{2\,3}_{2\,3} = \Gamma^{2\,1}_{2\,1} \qquad \Gamma^{3\,3}_{3\,3} = \Gamma^{3\,0}_{3\,0} + \Gamma^{2\,1}_{2\,1} + \Gamma^{1\,2}_{1\,2} \end{aligned} \tag{159}$$

where $\epsilon = \pm 1$ and θ, η are free parameters, give us the two-parameter integrable models introduced in [38]. It is interesting to mention that these two models contain as special cases the Hubbard model, the Essler–Korepin–Schoutens model, as well as the q-deformation of the extended Hubbard models introduced in [48, 50].

In all cases the spectral parameters $(k_1, \ldots, k_n)$ are going to be fixed by the cyclic property of the trace. The relations (145) and (148) imply that any amplitude of the matrix product ansatz (136) and (137) are proportional to $\mathrm{Tr}\big[Y_{k_1}^{(Q_1)} \cdots Y_{k_n}^{(Q_n)} E^L \Omega_P\big]$. Successive applications of the commutation relations (145), (140) and (138) give us the equations that will fix the spectral parameters. These equations are obtained in a similar way as in the last section by choosing $s_1 = s_2 = 1$ in (125)–(132) and are given by

$$\mathrm{e}^{-\mathrm{i}k_j L} = \prod_{l=1}^{n} S_{2\,2}^{2\,2}(k_l, k_j) \prod_{l=1}^{n_1} \frac{S_{2\,2}^{2\,2}\big(k_j, k_l^{(1)}\big)}{S_{2\,1}^{2\,1}\big(k_j, k_l^{(1)}\big)} \tag{160}$$

where the unknown parameters $k_l^{(1)}$ $(l = 1, \ldots, n)$ are fixed by

$$\prod_{l=1}^{n} \frac{S_{2\,1}^{2\,1}\big(k_l, k_j^{(1)}\big)}{S_{2\,2}^{2\,2}\big(k_l, k_j^{(1)}\big)} = \prod_{l=1}^{n_1} \frac{S_{2\,2}^{2\,2}\big(k_j^{(1)}, k_l^{(1)}\big)}{S_{1\,1}^{1\,1}\big(k_l^{(1)}, k_j^{(1)}\big)} \frac{S_{2\,1}^{2\,1}\big(k_l^{(1)}, k_j^{(1)}\big)}{S_{2\,1}^{2\,1}\big(k_j^{(1)}, k_l^{(1)}\big)} \tag{161}$$

for $j = 1, \ldots, n_1$.

6. The generalized XXZ chain in an open chain

Differently from the former sections where the quantum chains are all defined on periodic lattices we consider here the formulation of the matrix product ansatz in an open chain. We consider the generalized XXZ chain defined in section 2 in an open lattice with diagonal z-magnetic fields acting only at the surface points. The Hamiltonian we want to solve is given by

$$H_s = -\mathcal{P}_s \left\{ \frac{1}{2} \sum_{i=1}^{L-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right) + h_L \sigma_1^z + h_R \sigma_L^z \right\} \mathcal{P}_s \tag{162}$$

where $\sigma^x, \sigma^y, \sigma^z$ are spin-$\frac{1}{2}$ Pauli matrices, Δ the anisotropy and h_L, h_R are the magnetic fields acting at the end points 1 and L, respectively. The projector $\mathcal{P}_s$, as in section 2, projects out from the associated Hilbert space the configurations where any two up spins are at distances smaller than s $(s = 1, 2, \ldots)$. The choice $h_1 = h_L = 0$ corresponds to the free boundary case.

The exact solution of (162) for the case $s = 1$ was obtained through the coordinate Bethe ansatz in [39] and through the quantum inverse scattering method in [51]. It is also interesting to mention that, in the case where $s = 1$, the Hamiltonian (162) is $SU(2)_q$-invariant if the anisotropy and surface fields are related by $\Delta = (q + 1/q)/2$, $h_L = q$, $h_R = 1/q$.

Since the Hamiltonian (162) commutes with the total spin operator $S^z = \sum_{i=1}^{L} \sigma_i^z$ the number of up spins n is a good quantum number. We want to solve the eigenvalue equation

$$H|\Psi_n\rangle = \varepsilon_n |\Psi_n\rangle \tag{163}$$

where

$$|\Psi_n\rangle = \sum f(x_1, \ldots, x_n) |x_1, \ldots, x_n\rangle. \tag{164}$$

Here $x_1, \ldots, x_n$ denote the configurations of the up spins on the chain, and the summation extends over all sets of n increasing integers satisfying

$$x_1 \geqslant 1 \qquad x_n \leqslant L \qquad x_{i+1} \geqslant x_i + s \qquad (i = 1, \ldots, n-1). \tag{165}$$

As in section 2 in order to formulate our matrix product ansatz we associate the matrices E and A with the sites occupied by down and up spins, respectively[6]. Our ansatz asserts that any amplitude in (164) is given by

$$f(x_1, \ldots, x_n) = E^{x_1-1} A E^{x_2-x_1-1} A \cdots A E^{x_n-x_{n-1}-1} A E^{L-x_n}. \tag{166}$$

Actually E and A are abstract operators with an associative product. A well-defined eigenfunction (164) is obtained, apart from a normalization, if all the amplitudes are related uniquely. In order to obtain the solutions through the ansatz (166) let us consider initially the cases of small values of n.

$n = 1$. For one up spin the eigenvalue equation (163) gives us three types of relations depending on whether the corresponding configuration has the up spin at $x = 2, \ldots, L-1$ or at the boundaries $x = 1$ and $x = L$, namely

$$\begin{aligned} \varepsilon_1 E^{x-1} A E^{L-x} &= -E^{x-1}(E^{-1}AE + EAE^{-1})E^{L-x} \\ &\quad - \tfrac{1}{2}[(L-5)\Delta + h_L + h_R]E^{x-1}AE^{L-x} \qquad x = 2, \ldots, L-1 \end{aligned} \tag{167}$$

$$\varepsilon_1 A E^{L-1} = -EAE^{L-2} - \tfrac{1}{2}[(L-3)\Delta - h_L + h_R]AE^{L-1} \tag{168}$$

$$\varepsilon_1 E^{L-1} A = -E^{L-2}AE - \tfrac{1}{2}[(L-3)\Delta + h_L - h_R]E^{L-1}A. \tag{169}$$

The solution of all these equations is obtained by identifying the matrix A as composed by two other matrices B_k, C_k depending on a single spectral parameter k, i.e.,

$$A = (B_k - C_k)E^{2-s} \tag{170}$$

with the following commutation relation with the matrix E:

$$EB_k = \mathrm{e}^{\mathrm{i}k} B_k E \qquad EC_k = \mathrm{e}^{-\mathrm{i}k} C_k E. \tag{171}$$

Substituting (170) in (167) and using (171) we obtain the energy in terms of the spectral parameter k,

$$\varepsilon_1 = -2\cos(k) - \tfrac{1}{2}[(L-5)\Delta + h_L + h_R]. \tag{172}$$

Inserting (172), (170) in (168), (169) and using (171) we obtain the following algebraic relations:

$$\alpha(k)\,\mathrm{e}^{-\mathrm{i}k} B_k - \alpha(-k)\,\mathrm{e}^{\mathrm{i}k} C_k = 0 \tag{173}$$

$$\beta(-k)\,\mathrm{e}^{\mathrm{i}k} B_k - \beta(k)\,\mathrm{e}^{-\mathrm{i}k} C_k = 0 \tag{174}$$

where $\alpha(k)$ and $\beta(k)$ are given by

$$\alpha(k) = 1 + (h_L - \Delta)\,\mathrm{e}^{\mathrm{i}k} \qquad \beta(k) = [1 + (h_R - \Delta)\,\mathrm{e}^{\mathrm{i}k}]\exp(-\mathrm{i}k(L+1)). \tag{175}$$

The compatibility of the relations (173) and (174) fixes the spectral parameter

$$\frac{\alpha(k)\beta(k)}{\alpha(-k)\beta(-k)} = 1. \tag{176}$$

$n = 2$. The eigenvalue equation produces now four types of relations depending on the relative location x_1, x_2 of the up spins. The amplitudes related to the configurations where $x_1 > 1, L > x_2 > x_1 + s$ give us

$$\begin{aligned} \epsilon_2 E^{x_1-1} A E^{x_2-x_1-1} A E^{L-x_2} &= -E^{x_1-1}(E^{-1}AE + EAE^{-1})E^{x_2-x_1-1}E^{L-x_2} \\ &\quad - E^{x_1-1}AE^{x_2-x_1-1}(E^{-1}AE + EAE^{-1})E^{L-x_2} \\ &\quad - \tfrac{1}{2}[(L-9)\Delta + h_L + h_R]E^{x_1-1}AE^{x_2-x_1-1}AE^{L-x_2} \end{aligned} \tag{177}$$

[6] Differently from the solution on the periodic lattice (see (4) in section 2) in the present case it is not necessary to define the matrices A with the superscript s.

while the configurations where the particles are at the matching conditions $x_1 > 1, L \mid x_2 = x_1 + S$ produce the relation

$$\epsilon_2 E^{x_1-1} A E^{s-1} A E^{L-x_1-s} = -E^{x_1-2} A E^{s} A E^{L-x_1-s} - E^{x_1-1} A E^{s} A E^{L-x-s-1} - \tfrac{1}{2}[(L-5)\Delta + h_L + h_R] E^{x_1-1} A E^{s-1} A E^{L-x-s}. \tag{178}$$

Lastly the amplitudes where one of the particles is at one end point give us the following relations. For $x_1 = 1, L > x_2 > x_1 + s$,

$$\epsilon_2 A E^{x_2-2} A E^{L-x_2} = -E A E^{x_2-3} A E^{L-x_2} - A E^{x_2-3} A E^{L-x_2+1} - A E^{x_2-1} A E^{L-x_2-1} - \tfrac{1}{2}[(L-7)\Delta - h_L + h_R] A E^{x_2-2} A E^{L-x_2} \tag{179}$$

and for $1 < x_1 < L - s, x_2 = L$

$$\epsilon_2 E^{x_1-1} A E^{L-x_1-1} A = -E^{x_1-2} A E^{L-x_1} A - E^{x_1} A E^{L-x_1-2} A - E^{x_1-1} A E^{L-x_1-2} A E - \tfrac{1}{2}[(L-7)\Delta + h_L - h_R] E^{x_1-1} A E^{L-x_1-1} A. \tag{180}$$

The relations coming from the amplitudes where the particles are located at ($x_1 = 1$, $x_2 = 1 + s$) or ($x_1 = L - s, x_2 = s$) are satisfied by the solutions of (177)–(180). The solution of (177)–(180) is obtained by a generalization of (170), where we identify the matrix A as composed of $n = 2$ pairs of spectral parameter matrices $\{B_{k_j}, C_{k_j}\}$,

$$A = \sum_{j=1}^{n} \left(B_{k_j} - C_{k_j}\right) E^{2-s} \tag{181}$$

obeying the following commutation relations

$$\begin{aligned} &E B_{k_j} = \mathrm{e}^{\mathrm{i}k_j} B_{k_j} E \qquad E C_{k_j} = \mathrm{e}^{-\mathrm{i}k_j} C_{k_j} E \\ &B_{k_j}^2 = C_{k_j}^2 = B_{k_j} C_{k_j} = C_{k_j} B_{k_j} = 0 \qquad j = 1, \ldots, n. \end{aligned} \tag{182}$$

Inserting (181) into (177) we obtain the energy ϵ_2 in terms of the unknown spectral parameters k_1, k_2;

$$\epsilon_n = -2 \sum_{j=1}^{n} \cos(k_j) - \frac{1}{2}\left[(L - 1 - 4n)\Delta + h_L + h_R\right] \tag{183}$$

where $n = 2$. The 'bulk' relations (178) give us, after using (181) and (182), the following algebraic relations among the matrices $\{B_{k_j}, C_{k_j}\}$:

$$\begin{aligned} &a(k_l, k_j) B_{k_j} B_{k_l} + a(k_j, k_l) B_{k_l} B_{k_j} = 0 \qquad a(-k_l, -k_j) C_{k_j} C_{k_l} + a(-k_j, -k_l) C_{k_l} C_{k_j} = 0 \\ &a(-k_l, k_j) B_{k_j} C_{k_l} + a(k_j, -k_l) C_{k_l} B_{k_j} = 0 \qquad j, l = 1, \ldots, n \quad (j \neq l) \end{aligned} \tag{184}$$

with $n = 2$ and

$$a(k, k') = 1 - 2\Delta\, \mathrm{e}^{-\mathrm{i}k'} + \mathrm{e}^{-\mathrm{i}(k+k')}. \tag{185}$$

Using (181) and (182) in (179) and (180) gives us the following additional relations:

$$\alpha(k_j)\, \mathrm{e}^{-\mathrm{i}k_j} B_{k_j} B_{k_l} - \alpha(-k_j)\, \mathrm{e}^{\mathrm{i}k_j} C_{k_j} B_{k_l} = 0 \tag{186}$$

$$\alpha(-k_j)\, \mathrm{e}^{\mathrm{i}k_j} C_{k_j} C_{k_l} - \alpha(k_j)\, \mathrm{e}^{-\mathrm{i}k_j} B_{k_j} C_{k_l} = 0 \tag{187}$$

$$\beta(-k_l)\, \mathrm{e}^{-\mathrm{i}k_l s} B_{k_j} B_{k_l} - \beta(k_l)\, \mathrm{e}^{\mathrm{i}k_l s} B_{k_j} C_{k_l} = 0 \tag{188}$$

$$\beta(k_l)\, \mathrm{e}^{\mathrm{i}k_l s} C_{k_j} C_{k_l} - \beta(-k_l)\, \mathrm{e}^{-\mathrm{i}k_l s} C_{k_j} B_{k_l} = 0 \qquad j, l = 1, \ldots, n \quad (j \neq l) \tag{189}$$

with $n = 2$ and

$$\alpha(k) = 1 + (h_L - \Delta)\, \mathrm{e}^{\mathrm{i}k} \qquad \beta(k) = [1 + (h_R - \Delta)\, \mathrm{e}^{\mathrm{i}k}]\, \mathrm{e}^{-\mathrm{i}k(L+1)}. \tag{190}$$

The up to now free spectral parameters k_1 and k_2 are going to be fixed by imposing the compatibility of the algebraic relations (184), (186)–(189). Using successively (186), (184), (188) and (184) we obtain

$$\frac{\alpha(k_j)\beta(k_j)}{\alpha(-k_j)\beta(-k_j)}\,\mathrm{e}^{2\mathrm{i}k_j(s-1)} = \frac{\mathcal{B}(-k_j,k_l)}{\mathcal{B}(k_j,k_l)} \qquad j=1,2 \quad l\neq j \tag{191}$$

where

$$\mathcal{B}(k,k') = a(k,k')a(k',-k). \tag{192}$$

General n. The eigenvalue equation applied to the components corresponding to the configurations where no collisions exist ($x_{i+1} > x_i + s$; $i = 1,\ldots,n-1$) produces a generalization of the relation (177) that is solved by identifying, as in (181), the matrix A as composed by n pairs of spectral parameter dependent matrices $\{B_k, C_k\}$ satisfying the algebraic relations (182). In terms of the spectral parameters $k_1,\ldots,k_n$ the energy is given by (183). The eigenvalue equation (163) when applied to the components related to the configurations where two particles at x_i and x_{i+1} are at the 'colliding' positions $L > x_{i+1} = x_i + s > s$ give us the relations (184) for $j \neq l$, $j,l = 1,\ldots,n$. The configurations where we have a particle at the end points $x_1 = 1$ or $x_n = L$ give us the additional relations

$$\begin{aligned}
&\left[B_{k_{i_1}}B_{k_{i_2}}\alpha\left(k_{i_1}\right)\mathrm{e}^{-\mathrm{i}k_{i_1}} - C_{k_{i_1}}B_{k_{i_2}}\alpha\left(-k_{i_1}\right)\mathrm{e}^{\mathrm{i}k_{i_1}}\right]X_{k_{i_3}}\cdots X_{k_{i_n}} = 0\\
&\left[C_{k_{i_1}}C_{k_{i_2}}\alpha\left(-k_{i_1}\right)\mathrm{e}^{ik_{i_1}} - B_{k_{i_1}}C_{k_{i_2}}\alpha\left(k_{i_1}\right)\mathrm{e}^{-\mathrm{i}k_{i_1}}\right]X_{k_{i_3}}\cdots X_{k_{i_n}} = 0\\
&X_{k_{i_1}}\cdots X_{k_{i_{n-2}}}\left[B_{k_{i_{n-1}}}B_{k_{i_n}}\beta\left(-k_{i_n}\right)\mathrm{e}^{\mathrm{i}k_{i_n}[(n-1)(1-s)-1]}\right.\\
&\qquad\left. - B_{k_{i_{n-1}}}C_{k_{i_n}}\beta\left(k_{i_n}\right)\mathrm{e}^{-\mathrm{i}k_{i_n}[(n-1)(1-s)-1]}\right] = 0\\
&X_{k_{i_1}}\cdots X_{k_{i_{n-2}}}\left[C_{k_{i_{n-1}}}C_{k_{i_n}}\beta\left(k_{i_n}\right)\mathrm{e}^{-\mathrm{i}k_{i_n}[(n-1)(1-s)-1]}\right.\\
&\qquad\left. - C_{k_{i_{n-1}}}B_{k_{i_n}}\beta\left(-k_{i_n}\right)\mathrm{e}^{\mathrm{i}k_{i_n}[(n-1)(1-s)-1]}\right] = 0
\end{aligned} \tag{193}$$

where $i_1,\ldots,i_n$ is an arbitrary permutation of the integers $1,2,\ldots,n$, and X_{k_j} denotes a matrix B_{k_j} or C_{k_j}. It is interesting to observe that while the algebraic relations (184) only relate two products of two matrices the relations (193) relate the product of n matrices.

The matrix product ansatz (166) works only if all the amplitudes of the eigenfunction (164) are uniquely related. In fact the algebraic relations (182), (184) and (193) enable us to show that any amplitude given by the ansatz (166) is proportional to the matrix product $B_{k_1}\cdots B_{k_n}E^{L-n}$. The spectral parameters $k_1,\ldots,k_n$ are fixed by imposing the compatibility of the algebraic relations (184) and (193). For any $j = 1,\ldots,n$ we have

$$\begin{aligned}
B_{k_1}\cdots B_{k_j}\cdots B_{k_n} &= \left[\prod_{l=j+1}^{n}\frac{a(k_j,k_l)}{a(k_l,k_j)}\right]\frac{\beta(k_j)}{\beta(-k_j)}\\
&\quad\times \mathrm{e}^{-2\mathrm{i}k_j[(n-1)(1-s)-1]}B_{k_1}\cdots B_{k_{j-1}}B_{k_{j+1}}\cdots B_{k_n}C_{k_j}\\
&= \mathrm{e}^{-2\mathrm{i}k_j(n-1)(1-s)}\frac{\alpha(k_j)\beta(k_j)}{\alpha(-k_j)\beta(-k_j)}\\
&\quad\times\prod_{l=1,l\neq j}^{n}\frac{a(k_j,k_l)a(k_l,-k_j)}{a(k_l,k_j)a(-k_j,k_l)}B_{k_1}\cdots B_{k_j}\cdots B_{k_n}
\end{aligned} \tag{194}$$

that give us

$$\frac{\alpha(k_j)\beta(k_j)}{\alpha(-k_j)\beta(-k_j)}\,\mathrm{e}^{2\mathrm{i}k_j(n-1)(s+1)} = \prod_{l=1,l\neq j}^{n}\frac{\mathcal{B}(-k_j,k_l)}{\mathcal{B}(k_j,k_l)} \qquad j=1,\ldots,n. \tag{195}$$

The eigenvalues are given by (183) with the spectral parameters obtained by the solutions of (195). Equations (183) and (195) generalize the known result for $s = 1$ derived in [39] through the coordinate Bethe ansatz.

7. Conclusions and generalizations

Formulations of a matrix product ansatz have been introduced over the years in order to describe the ground-state wavefunctions of some special quantum chains. In general only the ground-state wavefunction of these models is described by this ansatz and the quantum chain is not exactly integrable. An exception occurs in the formulation named dynamical matrix product ansatz where the matrices defining the ansatz are time dependent. In this case the full eigenspectra of some exactly integrable quantum chains related to stochastic model are derived. We have shown in this paper that a huge family of exactly integrable quantum chains, normally solved through the Bethe ansatz, can also be solved by an appropriate matrix product ansatz. In our formulation, independently if the quantum chain is related or not to a stochastic model, the matrices are time independent. Differently from the Bethe ansatz where the amplitudes of eigenfunctions are given by combinations of plane waves, in the matrix product ansatz these amplitudes are given by a product of matrices.

A necessary condition for the integrability of the model through the matrix product ansatz is the existence of an abstract algebra with an associative product among the matrices defining the ansatz. In addition, since the amplitudes of the eigenfunctions are related to these matrix products, the algebraic properties of the matrices should provide a single relation among any two matrix products appearing in the ansatz. These algebraic relations are obtained by imposing the eigenvalue equation and depend on the coupling constants defining the quantum chain.

We have shown the formulation of the matrix product ansatz for two classes of models: models with a single global conservation law ($U(1)$ symmetry) such as the XXZ chain (see sections 2 and 3), the spin-1 Fateev–Zamolodchikov model, and models with two conservation laws ($U(1) \times U(1)$ symmetry) such as the spin-1 Perk–Schultz models, the Hubbard models as well as the other models presented in sections 4 and 5.

Let us discuss initially the models with a single conservation law. The associativity of the algebra, in this case, is immediate since the structure constants defining the algebraic relations among the matrices in the ansatz are complex constant numbers. In the case of the XXZ chain the algebraic relations merged in the sector with two particles are enough to ensure the exact integrability of the quantum chain (see section 2). For the spin-1 models there appear additional relations involving the product of three and four matrices since the coupling constants defining the Hamiltonian connect up to four particles at nearest-neighbour sites. The generalization of the matrix product ansatz for higher spin models ($s > 1$), although we did not consider it in this paper, follows straightforwardly. For example for spin $s = 3/2$ models we should relate to the spins $-3/2$, $-1/2$, $1/2$ and $3/2$, in the S^z-basis, the matrices E, A, $BE^{-1}B$ and $CE^{-1}CE^{-1}C$, respectively. In this case algebraic relations involving the product of three, four, five and six matrices will appear.

In the case of models with two global conservation laws (sections 4 and 5) the structure constants defining the algebraic relations among the product of two matrices are also matrices (S matrix) and the associativity condition is equivalent to the famous Yang–Baxter relations for the S-matrices. In the case of the spin-1 Perk–Schultz model (see section 4) the associativity of the algebra is enough to ensure the exact integrability of the quantum chain, however in the case of the Hubbard model, as well as the other quantum chains presented in section 5, there appear additional relations among the product of three and four matrices.

Generalizations of the present matrix product ansatz to the cases where we have three or more conservation laws follow straightforwardly. As we have shown in this paper (see sections 2, 3, 4 and 6) our formulation of the matrix product ansatz allows the extension of several exact integrable models by including arbitrary hard-core effects, without destroying their exact integrability. Also it is interesting to mention that in the cases of exactly integrable Hamiltonians associated with stochastic models, as in [17, 16], since we can write all eigenfunctions in a matrix product formulation, our results imply that we can equivalently write at any time the probability distribution of the model in terms of a time-dependent matrix product ansatz as happens in the formulation of the dynamical matrix product ansatz.

Except in section 6, all the quantum chains considered in this paper are defined on a periodic lattice and the eigenenergies are fixed by the cyclic property of the trace of the product of matrices appearing in the ansatz. In section 6, we show how to formulate the matrix product ansatz in the case where the quantum chains are defined on open lattices. We derived the solution of the XXZ chain with magnetic fields at the endpoints of the lattice. The formulation of a matrix product ansatz for the other models of sections 3–5 with open boundary conditions that preserve their global symmetry is also possible.

Since the exactly integrable chains considered in this paper share the same eigenfunctions with a related two-dimensional vertex model the matrix product ansatz we formulated also provides a solution for these classical models. A quite interesting problem for the future concerns the formulation of the matrix product ansatz for the quantum chains with no global conservation law such as the XYZ model, the 8-vertex model or the case where the quantum chains are defined on open lattices with non-diagonal boundary fields.

In conclusion our results induce us to conjecture that a matrix product ansatz, along the lines presented in this paper, can be formulated for any exactly integrable quantum chain. The importance of this ansatz for the future, as shown in this paper, lies in its simplicity, allowing quite simple generalizations and the formulation of known exactly integrable models.

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